

Put Call Reversal and Semi-Static Hedging

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Summary

- Since this talk is about time reversal, it seems only appropriate that I begin this talk at its end.
- So in summarizing this paper, we showed how Put Call Reversal is the financial version of Newton's Third Law: For every option, there is an equal and opposite option.
- More specifically, given a European call written on a forward running stock price process S , there exists a European put written on some backward running stock price process \hat{S} , such that the call and the put have the same value functions.
- As we assumed that S is a Markov semi-martingale, the backward running stock price \hat{S} is also a Markov semi-martingale.
- As we assumed no arbitrage in the forward economy, there is also no arbitrage in the backward economy.
- Given deterministic interest rates r and dividend yields q , no arbitrage causes the risk-neutral mean of the stock price to grow at the relative rate of $r - q$. Hence, in the backward economy, the risk-neutral mean of the backward running stock price grows at $q - r$. In the backward economy, we have deterministic interest rates q and dividend yields r .

Put Call Reversal

- Let $C(t_0, S; T, K)$ be the value of a European call at some time $t_0 \geq 0$ given $S_{t_0} = S$ with maturity $T \geq t_0$ and strike $K \geq 0$.

- Consider some economy in which time runs backwards and in this economy, let $\hat{P}(\hat{t}, \hat{S}; \hat{T}, \hat{K})$ be the value of a European put at the reverse time \hat{t} given that the underlying backward running stock price $\hat{S}_{\hat{t}} = \hat{S}$ with maturity $\hat{T} \leq \hat{t}$ and strike \hat{K} .

- Then Put Call Reversal is:

$$C(t_0, S_0; T, K) = \hat{P}(T, K; t_0, S_0), \quad S_0, K \in \mathfrak{R}^+, T \geq t_0 \geq 0.$$

- The valuation date for the put is the call's maturity date and vice versa.
- The initial spot level for the put's underlying is the call's strike price and vice versa.
- The maturity date of the put is the call's valuation date and vice versa.
- The strike of the put is the call underlying's initial spot and vice versa.

Why is it True?

- In our 1 factor Markov setting with deterministic interest rates and dividend yields, we show that call values curve up in spot.
- Given this fact, the proof of PCR is exceedingly simple. By the sifting property of Dirac's delta function, we have:

$$C(t_0, S_0; T, K) = \int_0^\infty \delta(S - S_0) C(t_0, S; T, K) dS. \quad (1)$$

- Integrating (1) by parts twice gives:

$$C(t_0, S_0; T, K) = \int_0^\infty \frac{\partial^2}{\partial S^2} C(t_0, S; T, K) (S_0 - S)^+ dS, \quad (2)$$

since $C(t_0, 0; T, K) = C_s(t_0, 0; T, K) = 0$. Thus, options are valued by integrating their *gammas* against hockey sticks!

- As $C_{ss}(t_0, S; T, K)$ is nonnegative and integrable in S , we can interpret it as the price at T of a butterfly spread in a reverse time economy. After all, $\lim_{T \rightarrow t_0} C_{ss}(t_0, S; T, K) = \delta(K - S)^+$.
- This butterfly spread has strike S and maturity t_0 , and is written on an asset which at the initial time T has initial price K .
- (2) then values a *put* maturing at t_0 and struck at S_0 . The put is written on some backward running process which is conditioned to start at time T at the level K .

Forward Stock Price Dynamics

- In the forward economy, no arbitrage allowed us to define a measure Q and an associated standard Brownian motion B such that the forward running stock price S solves the following stochastic differential equation (SDE):

$$\frac{dS_t}{S_{t-}} = [r(t) - q(t)]dt + \sigma(S_{t-}, t)dB_t \quad (3)$$
$$+ \int_{-\infty}^{\infty} (e^x - 1)(dx, dt) - \nu(x, t)dxdt], \text{ for all } t \in [0, \bar{T}],$$

where the initial stock price is known.

- For the above process, the call's gamma $C_{ss}(t, S; T, K)$ is non-negative and integrable in S . If we normalize it into a transition density, it describes a backward running process \hat{S} conditioned to start at K at time T .
- The call gamma is used to relate the coefficients of the SDE governing this backward running process to the coefficients of the SDE governing the forward running process.

Backward Stock Price Dynamics

- Recall the forward time stock price dynamics:

$$\begin{aligned} \frac{dS_t}{S_{t-}} &= [r(t) - q(t)]dt + \sigma(S_{t-}, t)dB_t \\ &+ \int_{-\infty}^{\infty} (e^x - 1)[\mu(dx, dt) - \nu(x, t)dxdt], \text{ for all } t \in [0, \bar{T}], \end{aligned} \quad (4)$$

- Let $\hat{t} \equiv \bar{T} - t$ denote reversed time.
- The reversed process induced by the call's gamma and underlying the put solves the SDE:

$$\begin{aligned} \frac{d\hat{S}_{\hat{t}}}{\hat{S}_{\hat{t}-}} &= [q(\bar{T} - \hat{t}) - r(\bar{T} - \hat{t})]d\hat{t} + \sigma(\hat{S}_{\hat{t}-}, \bar{T} - \hat{t})d\hat{B}_{\hat{t}} \\ &+ \int_{-\infty}^{\infty} (e^{\hat{x}} - 1)[\hat{\mu}(d\hat{x}, d\hat{t}) - e^{-\hat{x}}\nu(-\hat{x}, \bar{T} - \hat{t})d\hat{x}d\hat{t}], \end{aligned} \quad (5)$$

for all $\hat{t} \in [0, \bar{T}]$.

- Thus, the volatility is the same function of spot but flipped in time. The arrival rate of a jump of size x in the forward economy is multiplied by e^x to obtain the arrival rate of a jump of size $\hat{x} \equiv -x$ in the backward economy.

PCR vs. PCE

- Suppose that interest rates and dividend yields are constant and let $C(S_0, q, t_0; K, r, T)$ & $P(S_0, q, t_0; K, r, T)$ be European call and put values at spot S_0 , div yield q , time t_0 , strike K , riskfree rate r , & maturity $T \geq t_0$. Then PCR implies:

$$C(S_0, q, t_0; K, r, T) = P(K, r, T; S_0, q, t_0).$$

- Put Call Equivalence (PCE) implies that a put on one currency is the same contract as a call on the other. PCE looks like PCR, but PCR is not a consequence of PCE. To see why, note that under no jumps, the stock price dynamics under Q are:

$$\frac{dS_t}{S_t} = [q(t) - r(t)]dt + \sigma(S_t, t)dB_t.$$

- PCE is based on inversion in the underlying and a measure change. Letting $\tilde{S}_t \equiv \tilde{S}_0 \frac{S_0}{S_t}$, then one can change $Q \rightarrow \tilde{Q}$ so:

$$\frac{dS_t}{S_t} = [r(t) - q(t)]dt + \sigma\left(S_0 \frac{\tilde{S}_0}{\tilde{S}_t}, t\right)d\tilde{B}_t$$

- The corresponding dynamics for \hat{S} under \hat{Q} are:

$$\frac{d\hat{S}_{\hat{t}}}{\hat{S}_{\hat{t}}} = [q(\bar{T} - \hat{t}) - r(\bar{T} - \hat{t})]d\hat{t} + \sigma(\hat{S}_{\hat{t}}, \hat{t})d\hat{B}_{\hat{t}}.$$

- Hence the volatility functions are different.

Applications: Interpreting Delta & Gamma

- Put Call Reversal is $C(t_0, S_0; T, K) = \hat{P}(T, K; t_0, S_0)$.

- Differentiating PCR w.r.t. S_0 implies:

$$\frac{\partial}{\partial S} C(t_0, S_0; T, K) = \frac{\partial}{\partial \hat{K}} \hat{P}(\bar{T} - T, K; \bar{T} - t_0, S_0). \quad (6)$$

- Thus, the delta of the call on the forward running process is the first strike derivative of the put on the reversed process. The latter is just the q -discounted probability that the reversed process finishes below S_0 when it starts from K .

- Differentiating (6) w.r.t. S_0 implies:

$$\frac{\partial^2}{\partial S^2} C(t_0, S_0; T, K) = \frac{\partial^2}{\partial \hat{K}^2} \hat{P}(\bar{T} - T, K; \bar{T} - t_0, S_0).$$

- Thus, the gamma of the call on the forward running process is the second strike derivative of the put on the reversed process. The latter is just the q -discounted PDF under \hat{Q} of the event that the reversed process finishes at S_0 when it starts at K .

- As discounted probabilities are always nonnegative, so is a call's delta and gamma. Furthermore, the call's gamma must integrate over initial prices: $\int_0^\infty C_{ss}(S, t) dS = e^{-\int_t^T q(u) du}$.

Horizontal Semi-Static Hedging

- In the reverse time economy, the put's value at its initial date T is just the q -discounted expectation of the value it will have at any intermediate date $\hat{t} = u$. Thus $\hat{P}(T, K; t_0, S_0) =$

$$e^{-\int_u^T q(v)dv} \int_0^\infty \hat{P}(u, L; t_0, S_0) \hat{Q}\{\hat{S}_u \in dL | \hat{S}_{\hat{t}_0} = K\}.$$

- Using Put Call Reversal twice, we have $C(t_0, S_0; T, K) =$

$$e^{-\int_u^T q(v)dv} \int_0^\infty C(t_0, S_0; u, L) \hat{Q}\{\hat{S}_u \in dL | \hat{S}_{\hat{t}_0} = K\}.$$

- Thus, a long dated call has the same value to the intermediate time as a portfolio of shorter dated calls of all strikes maturing at the intermediate time.
- The weight on a call of strike L is just the q -discounted probability that the reversed process transitions from K to L between $\hat{t} = T$ and $\hat{t} = u$. As we move forward in forward time from $t = t_0$, this weight does not change, so the hedge is static until the calls mature.

Vertical Semi-Static Hedging

- Suppose that $S_0 < K$ and consider some barrier $H \in (S_0, K)$. Then S must cross H for the call to finish in-the-money.
- Likewise, since the backward running stock price \hat{S} starts at K , it must cross H for the put to finish-in-the-money.
- In the reverse time economy, the put's value at its initial date T is just the q-discounted expectation of the value it will have at the first passage time $\hat{\tau}_H$ from K to H . Thus $\hat{P}(T, K; t_0, S_0) = \int_{t_0}^T e^{-\int_u^T q(v)dv} \int_0^H \hat{P}(u, L; t_0, S_0) \hat{Q}\{\tau_H \in du, \hat{S}_u \in dL | \hat{S}_{\hat{t}_0} = K\}$.
- Using Put Call Reversal twice, we have $C(t_0, S_0; K, T) = \int_{t_0}^T e^{-\int_u^T q(v)dv} \int_0^H C(t_0, S_0; u, L) \hat{Q}\{\tau_H \in du, \hat{S}_u \in dL | \hat{S}_{\hat{t}_0} = K\}$.
- Thus, a call has the same value initially as a portfolio of calls of all strikes $L < H$ and all nearer maturities.
- If the forward running stock price can not jump up, then the backward running stock price can not jump down, and the double integral collapses to a single integral over call maturities.

Diagonal Semi-Static Hedging

- The argument on the previous page just requires that the barrier be crossed with probability one in order that the option finish-in-the-money.
- As a result, we can generalize the argument to any time-dependent barrier which must be encountered by the forward-running stock price in order that the call finish in-the-money.
- The weight on each nearer-dated call is given by the probability that the reversed process crosses the call's strike for the first time at the call's maturity.
- Provided the call hasn't matured, this weight is static until the barrier is crossed.
- At the barrier crossing time, the entire portfolio is sold and the proceeds are just enough to buy the call with strike K and maturity T .
- Thus, time reversal allows us to make (semi-) dynamic option trading strategies transparent.

Introduction and Historical Research

- In keeping with our time reversal theme, it's now time to begin my talk.
- Fortunately, I'm referring to the reverse clock. So if my talk is a call, it has now matured (hopefully in-the-money).