American Options: A Comparison of Numerical Methods

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1 Introduction

The overwhelming majority of traded options are of American type. Yet their valuation, even in the standard case of a lognormal process for the underlying asset, remains a topic of active research. This situation stems from the nature of the solution which requires the determination of the optimal exercise strategy as well as the value of the option. In contrast the European option, which can only be exercised at its expiration date, has been valued by the celebrated Black–Scholes formula (Black & Scholes 1973) for the standard financial model.

Due to a lack of closed–form solutions to American option valuation problems, a vast array of approximation schemes has been advanced. The Broadie & Detemple article in this volume provides a summary of some experimental results. The present article is a detailed account of comparative experiments conducted with numerical schemes including the recent work of Carr & Faguet 1996.It is organized a follows: Section 2 reviews the basic Black–Scholes model, Section 3 presents the approximation approaches and Section 4 concludes with some benchmark comparisons.

2 The Standard Model

The proptotypical definition of an American option is that of a contract giving its holder the right to buy (call option) or sell (put option) one unit of an underlying security (e.g. stock) at a pre-arranged price K. This right can be exercised at any time before an expiration date T. In contrast, a European option can be exercised at the expiration date only.

In the standard model, also called the Black–Scholes/Merton environment (Black & Scholes 1973, Merton 1973), the market consists of the option, its underlying security, labelled the stock, and a riskless security, labelled the bond. This market is populated by equally informed traders who do not incur transaction costs, among other simplifying assumptions. At any time t an amount β_t in the bond will evolve according to the differential equation

$$d\beta_t = r\beta_t dt,$$

where r is the riskless rate of lending and borrowing. The randomness of the continuous stock price process $\{P_t\}$ is modelled as the geometric Brownian motion

$$P_t = P_0 e^{(\alpha-\delta-\sigma^2/2)t+\sigma\tilde{W}_t}, \tag{1}$$

where P_0 is the initial stock price, α the mean rate of the stock return over an infinitesimal interval, σ is the associated deviation (volatility) and δ the dividend rate paid by the stock. Here $\{\tilde{W}\}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$.

For this model there exists a probability measure Q_T equivalent to P such that P_t is a martingale in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q_T)$. By Girsanov's theorem

$$W_t = \frac{\alpha - r}{\sigma} t + \tilde{W}_t$$

is a standard Brownian motion in $(\Omega, \mathcal{F}, \{\mathcal{F}_\sqcup\}, \mathcal{Q}_\mathcal{T})$. Thus

$$P_t = P_0 e^{(r-\delta-\sigma^2/2)t+\sigma W_t},$$

and

$$d\left\{e^{-rt} P_t\right\} = e^{-rt} P_t \left(\sigma dW_t - \delta dt\right),$$

indicating that, in the absence of dividend, the discounted stock price $\{e^{-rt} P_t\}$ is martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q_T)$. For this property Q_T is sometimes called an 'equivalent martingale measure'. Note that if the dividend rate is non-negative then the discounted price process is a supermatingale under Q_T. For further details we refer the reader to Bensoussan (1984), Duffie (1988), Karatzas (1988) and Myneni (1992).

The characterization of American option valuation as an optimal stopping problem goes back at least to McKean (1965) who based his work on Samuelson's (1965) pricing model. However, it was not until Bensoussan (1984) and Karatzas (1988) that an arbitrage argument was provided. They show that the option price $U_t \equiv U(t, P_t)$ at time $t \in [0, T]$ is

$$U_t = \operatorname{ess} \sup_{\tau \in \mathcal{T}_{t,T}} E_{Q_T} \left(e^{-r(\tau-t)} f(P_\tau) | \mathcal{F}_t\right)$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times in $[t, T]$, and $f(x) = (K - x)^+$ $(f(x) = (x - K)^+)$ for a put (respectively a call) with exercise price K.

Under this optimal stopping formulation there exists, for the put, a function $t \mapsto \bar{P}_t$ in $\mathcal{C}[0, T)$, non-decreasing, and independent of the initial stock price, such that

$$\begin{aligned} &\text{if } P_t \leq \bar{P}_t \text{ then } U(t, P_t) = (K - P)^+, \\ &\text{if } P_t > \bar{P}_t \text{ then } U(t, P_t) > (K - P)^+. \end{aligned}$$

Symmetrically, for a call there corresponds a function $t \mapsto \underline{P}(t)$ in $\mathcal{C}[0, T)$, non-increasing and independent of the initial stock price, such that

$$\begin{aligned} &\text{if } P_t \leq \underline{P}(t) \text{ then } U(t, P_t) > (P_t - K)^+, \\ &\text{if } P_t > \underline{P}(t) \text{ then } U(t, P_t) = (P_t - K)^+. \end{aligned}$$

We refer to Friedman (1975), Van Moerbeke (1976), Jacka (1991) for the supporting arguments for the above statements.

Given that the stochastic dynamics are governed by a Brownian motion the determination of the critical (or exercise) price $\bar{P}(t)$ or $\underline{P}(t)$ rests on solving a free–boundary problem for the heat equation.

3 Approximation Methods

There are no known closed–form solutions to the above optimal stopping problems except for the infinite horizon case (McKean 1965). Many authors have therefore followed the approximate solution path. Recent efforts have indicated directions in which some closed–form expressions can be obtained for the optimal exercise boundary (AitSahlia 1995, AitSahlia & Lai 1996) or the value function (Carr & Faguet 1996).

Carr & Faguet (1996) present an approach that is comparable to the extant methods. They investigate the analytic method of lines as a way to evaluate American options. In common with the numerical method of lines (Meyer 1979, Meyer & Van der Hoek 1994), this approach discretizes the time derivative in the Black–Scholes partial differential equation. In contrast with the numerical method of lines, the authors then solve the resulting series of ordinary differential equations analytically. This analytic solution is protected against any truncation and spatial discretization error implicit in numerical schemes. A notable feature of this approach is that the critical stock price at each time step may be solved explicitly, so long as the underlying stock either pays no dividend or has constant continuous payout. In common with several other papers (e.g. Geske & Johnson 1984, Subrahmanyam & Yu 1993, Breen 1991), the authors use Richardson extrapolation to speed up the computation of the solution.

In the next section this approach is compared against the following numerical schemes:

1. the standard binomial model (Cox, Ross & Rubinstein 1979, Rendleman & Barrter 1979) based on the archetypical random walk approximation to the Brownian motion,
2. the 'accelerated' binomial model (Breen 1991), which uses Richardson's extrapolation technique (Marchuk & Shaidurov 1983) to reduce the number of steps,
3. the BBS binomial method of Broadie & Detemple (1996), which is based on using the Black–Scholes formula 1 step before expiration,
4. the capped option formulae of Broadie & Detemple (1996), which result in a lower bound labelled LBA and a weighted average of lower and upper bounds labelled LUBA,
5. the trinomial method (Parkinson 1977, Kamrad & Ritchken 1991),
6. the quadratic formula (McMillan 1986, Barone-Adesi & Whalley 1987), which seeks an approximate solution to the Black–Scholes PDE by neglecting a quadratic term for the exercise premium,
7. the regression formula (Johnson 1983) where, as in the LUBA approximation above, the American option value is regressed against lower and upper bounds,
8. the exponential formula (Omberg 1987 and Chesney 1989), where the value function is approximated on the assumption that the optimal stopping boundary is of the exponential form,

9. the implicit finite-difference scheme (Brennan & Schwartz 1977), which is classically applied to the variational inequality formulation of the free–boundary problem,

10. the two–point GJ formula with Richardson extrapolation (see Geske & Johnson 1984) and with exponential extrapolation (Ho, Stapleton & Subrahmanyam 1994), where an American option is viewed as a compound option (option on an option), exerciseable only at a series of discrete dates,

11. the modified GJ approach (Bunch & Johnson 1992),

12. the numerical integral representation of the early exercise premium, with or without Richardson extrapolation (Subrahmanyam & Yu 1993).

4 Some Numerical Comparisons

Because of its simplicity and convergence the binomial method has attracted the most attention and been modified into a number of variants which resulted in improved accuracy, sometimes at the expense of speed. The results of Table 1b support the empirical reliance of practitioners on the penny accuracy of 200–step binomial methods. Therefore, for all practical purposes we have found it reasonable to use the average of the 1,000 and 1,001 time binomial values as 'exact'. Against this benchmark we use four accuracy criteria, the most prominent of which we deem is the mean squared relative error as defined Broadie & Detemple (in this volume).

The speed of each of these methods is measured in calculation time (in seconds) per put.

The results of preliminary experiements appear in line with those of Broadie & Detemple. Our results are displayed in Tables 2 and 3, and are further summarized in Figures 1 and 2. With no dividend, the five dominant approaches, ranked in terms of increasing computation time and accuracy are:

1. Johnson's regression formula,

2. Omberg's exponential formula,

3. Method of lines, especially the modified 3–point extrapolation,

4. Broadie & Detemple's lower bound and their approximation based on both bounds,

5. Modified binomial method.

We also considered options on dividend–paying stocks with an average of three years to maturity and obtained broadly similar results. However, of the five dominant approaches above, the first two are not defined for positive dividends, so only the latter three dominate in this case.

In short, the method of lines increases in accuracy as maturity increases, but decreases in speed as dividends are added. The accuracy increase is due to better approximation of the exercise boundary, while the speed decrease is due to the required numerical solution of the critical stock price.

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Table 1a: Convergence of Method of Lines with Richardson Extrapolation
$S = 100$, $K = 100$, $T = 1$, $r = 0.1$, $\delta = 0$, $\sigma = 0.3$

Number of Steps n or Points N	Unextrapolated Put Value $P^{(n)}$	Extrapolated Put Value $P^{1:N}$
1	7.0405	7.0405
2	7.6175	8.1946
3	7.8353	8.3089
4	7.9505	8.3257
5	8.0220	8.3311
6	8.0709	8.3333
7	8.1065	8.3345
8	8.1335	8.3353
9	8.1548	8.3358
10	8.1720	8.3362
11	8.1862	8.3365
12	8.1981	8.3367
13	8.2082	8.3369
14	8.2169	8.3370
15	8.2246	8.3371

Table 1b: Convergence of the Binomial Model
$S = 100$, $K = 100$, $T = 1$, $r = 0.1$, $\delta = 0$, $\sigma = 0.3$

N	Even Value	N	Odd Value	Average
50	8.315025	51	8.367897	8.341461
100	8.325495	101	8.352275	8.338885
150	8.329729	151	8.347580	8.338655
200	8.331951	201	8.345306	8.338628
250	8.333084	251	8.343817	8.338450
300	8.333922	301	8.342853	8.338388
350	8.334424	351	8.342078	8.338251
400	8.334808	401	8.341514	8.338161
450	8.335136	451	8.341099	8.338118
500	8.335404	501	8.340771	8.338087
550	8.335620	551	8.340500	8.338060
600	8.335805	601	8.340278	8.338041
650	8.335954	651	8.340083	8.338018
700	8.336085	701	8.339919	8.338002
750	8.336199	751	8.339779	8.337989
800	8.336293	801	8.339650	8.337971
850	8.336372	851	8.339532	8.337952
900	8.336446	901	8.339429	8.337938
950	8.336520	951	8.339347	8.337933
1000	8.336577	1001	8.339263	8.337920
1050	8.336627	1051	8.339185	8.337906
1100	8.336678	1101	8.339120	8.337899
1150	8.336721	1151	8.339057	8.337889
1200	8.336759	1201	8.338998	8.337879
1250	8.336799	1251	8.338948	8.337873
1300	8.336834	1301	8.338900	8.337867
1350	8.336865	1351	8.338854	8.337860
1400	8.336896	1401	8.338815	8.337855
1450	8.336920	1451	8.338773	8.337847
1500	8.336947	1501	8.338738	8.337843
1550	8.336970	1551	8.338704	8.337837
1600	8.336993	1601	8.338672	8.337832
1650	8.337014	1651	8.338643	8.337829
1700	8.337033	1701	8.338613	8.337823
1750	8.337053	1751	8.338589	8.337821
1800	8.337070	1801	8.338563	8.337816
1850	8.337088	1851	8.338540	8.337814
1900	8.337103	1901	8.338517	8.337810
1950	8.337119	1951	8.338497	8.337808
2000	8.337133	2001	8.338476	8.337804

Table 2a: Short Term Put Values, Strike Price = $100, No Dividend

Stock Price in $	Time to Exp'n	Vol-ati-lity	Risk free Rate	Average Bin'l, $N =$ 1,000	Method of Lines 3 pts	Quad-ratic Formula	Regres-sion Formula
80	0.500	40	6	21.6059	21.6257	21.5077	21.4678
85	0.500	40	6	18.0374	18.0402	17.9606	17.9802
90	0.500	40	6	14.9187	14.9190	14.8673	14.9102
95	0.500	40	6	12.2314	12.2318	12.2033	12.2498
100	0.500	40	6	9.9458	9.9417	9.9376	9.9780
105	0.500	40	6	8.0281	8.0228	8.0334	8.0638
110	0.500	40	6	6.4352	6.4233	6.4508	6.4704
115	0.500	40	6	5.1265	5.1047	5.1488	5.1586
120	0.500	40	6	4.0611	4.0328	4.0875	4.0890
100	0.500	40	2	10.7742	10.7899	10.7613	10.7757
100	0.500	40	4	10.3450	10.3478	10.3317	10.3649
100	0.500	40	6	9.9458	9.9417	9.9376	9.9780
100	0.500	40	8	9.5716	9.5642	9.5711	9.6086
100	0.500	40	10	9.2195	9.2109	9.2275	9.2548
100	0.500	30	6	7.2117	7.2060	7.2052	7.2286
100	0.500	35	6	8.5782	8.5731	8.5706	8.6036
100	0.500	40	6	9.9458	9.9417	9.9376	9.9780
100	0.500	45	6	11.3127	11.3100	11.3043	11.3505
100	0.500	50	6	12.6778	12.6770	12.6697	12.7202
100	0.083	40	6	4.3860	4.3926	4.3785	4.3762
100	0.167	40	6	6.0688	6.0737	6.0576	6.0639
100	0.250	40	6	7.3090	7.3117	7.2964	7.3131
100	0.333	40	6	8.3201	8.3205	8.3077	8.3340
100	0.417	40	6	9.1849	9.1830	9.1740	9.2083
100	0.500	40	6	9.9458	9.9417	9.9376	9.9780
100	0.583	40	6	10.6280	10.6218	10.6235	10.6681
100	0.667	40	6	11.2482	11.2400	11.2480	11.2951
100	0.750	40	6	11.8176	11.8075	11.8225	11.8702
100	0.833	40	6	12.3447	12.3327	12.3553	12.4020
100	0.917	40	6	12.8357	12.8220	12.8525	12.8967
Sec's per Put				1.74871	0.00022	0.00075	0.00013
Mean Rel. Err.				0.0000	-0.0006	-0.0005	0.0024
Root Mean Sqr'd Err.				0.0000	0.0098	0.0269	0.0417
Mean Abs. Rel. Err.				0.0000	0.0009	0.0016	0.0033
Max Rel. Err.				0.0000	-0.0070	0.0065	0.0069

Table 2b: Short Term Put Values, Strike Price = $100, No Dividend

Stock Price in $	Time to Exp'n	Vol-ati-lity	Risk free Rate	Omberg Exp'l Formula	Broadie Detemple LBA	Broadie Detemple LUBA
80	0.500	40	6	21.5832	21.6174	21.6034
85	0.500	40	6	18.0186	18.0400	18.0346
90	0.500	40	6	14.9033	14.9167	14.9174
95	0.500	40	6	12.2189	12.2274	12.2311
100	0.500	40	6	9.9365	9.9421	9.9466
105	0.500	40	6	8.0198	8.0238	8.0279
110	0.500	40	6	6.4286	6.4317	6.4349
115	0.500	40	6	5.1212	5.1237	5.1261
120	0.500	40	6	4.0569	4.0592	4.0607
100	0.500	40	2	10.7699	10.7800	10.7743
100	0.500	40	4	10.3391	10.3435	10.3457
100	0.500	40	6	9.9365	9.9421	9.9466
100	0.500	40	8	9.5585	9.5595	9.5721
100	0.500	40	10	9.2030	9.2213	9.2190
100	0.500	30	6	7.2002	7.2018	7.2114
100	0.500	35	6	8.5680	8.5711	8.5785
100	0.500	40	6	9.9365	9.9421	9.9466
100	0.500	45	6	11.3040	11.3130	11.3141
100	0.500	50	6	12.6694	12.6826	12.6797
100	0.083	40	6	4.3843	4.3886	4.3855
100	0.167	40	6	6.0655	6.0702	6.0683
100	0.250	40	6	7.3041	7.3191	7.3088
100	0.333	40	6	8.3137	8.3188	8.3202
100	0.417	40	6	9.1770	9.1824	9.1853
100	0.500	40	6	9.9365	9.9421	9.9466
100	0.583	40	6	10.6174	10.6234	10.6293
100	0.667	40	6	11.2363	11.2426	11.2498
100	0.750	40	6	11.8046	11.8111	11.8195
100	0.833	40	6	12.3305	12.3374	12.3469
100	0.917	40	6	12.8204	12.8276	12.8382
Sec's per Put				0.00040	0.00154	0.00567
Mean Rel. Err.				-0.0010	-0.0002	0.0000
Root Mean Sqr'd Err.				0.0110	0.0048	0.0012
Mean Abs. Rel. Err.				0.0010	0.0004	0.0001
Max Rel. Err.				-0.0018	-0.0014	0.0002

Table 2c: Short Term Put Values, Strike Price = $100, No Dividend

Stock Price in $	Time to Exp'n	Vol-ati-lity	Risk free Rate	IFD $N = 200$ $M = 200$	Method of Lines 6 points	Ho et al. 2pt GJ	Geske Johnson 2 points	Chesney Exp'l Formula
80	0.500	40	6	21.5989	21.6022	21.7875	21.7733	22.7094
85	0.500	40	6	18.0274	18.0317	18.1473	18.1384	17.8090
90	0.500	40	6	14.9066	14.9116	14.9606	14.9554	14.7108
95	0.500	40	6	12.2179	12.2235	12.2220	12.2192	12.0530
100	0.500	40	6	9.9327	9.9384	9.9051	9.9037	9.8000
105	0.500	40	6	8.0146	8.0198	7.9708	7.9701	7.9123
110	0.500	40	6	6.4230	6.4273	6.3740	6.3736	6.3501
115	0.500	40	6	5.1160	5.1193	5.0684	5.0682	5.0672
120	0.500	40	6	4.0526	4.0549	4.0099	4.0098	4.0202
100	0.500	40	2	10.7626	10.7684	10.7399	10.7399	10.7379
100	0.500	40	4	10.3324	10.3379	10.2997	10.2993	10.2486
100	0.500	40	6	9.9327	9.9384	9.9051	9.9037	9.8000
100	0.500	40	8	9.5585	9.5645	9.5457	9.5420	9.3913
100	0.500	40	10	9.2066	9.2127	9.2145	9.2072	9.0134
100	0.500	30	6	7.2007	7.2063	7.1887	7.1863	7.0801
100	0.500	35	6	8.5662	8.5718	8.5457	8.5439	8.4389
100	0.500	40	6	9.9327	9.9384	9.9051	9.9037	9.8000
100	0.500	45	6	11.2985	11.3045	11.2651	11.2638	11.1618
100	0.500	50	6	12.6622	12.6688	12.6241	12.6230	12.5223
100	0.083	40	6	4.3773	4.3834	4.3705	4.3705	4.4103
100	0.167	40	6	6.0594	6.0647	6.0428	6.0427	6.0457
100	0.250	40	6	7.2987	7.3039	7.2759	7.2756	7.2467
100	0.333	40	6	8.3088	8.3141	8.2825	8.2819	8.2233
100	0.417	40	6	9.1727	9.1781	9.1448	9.1439	9.0632
100	0.500	40	6	9.9327	9.9384	9.9051	9.9037	9.8000
100	0.583	40	6	10.6142	10.6202	10.5884	10.5863	10.4590
100	0.667	40	6	11.2336	11.2399	11.2109	11.2080	11.0565
100	0.750	40	6	11.8021	11.8089	11.7840	11.7801	11.6039
100	0.833	40	6	12.3281	12.3357	12.3158	12.3108	12.1098
100	0.917	40	6	12.8179	12.8263	12.8126	12.8063	12.5804
Sec's per Put				0.16513	0.00127	0.00133	0.00133	0.02244
Mean Rel. Err.				-0.0013	-0.0007	-0.0034	-0.0036	-0.0112
Root Mean Sqr'd Err.				0.0127	0.0071	0.0537	0.0524	0.2480
Mean Abs. Rel. Err.				0.0013	0.0007	0.0045	0.0046	0.0149
Max Rel. Err.				-0.0021	-0.0015	-0.0126	-0.0126	0.0511

Table 2d: Short Term Put Values, Strike Price = $100, No Dividend

Stock Price in $	Time to Exp'n	Vol- ati- lity	Risk free Rate	Method of Lines 9 points	Bunch Johnson Mod. GJ	Integral Method $N = 6$	Sub'n Yu 3 points
80	0.500	40	6	21.6044	21.8225	21.6516	21.7099
85	0.500	40	6	18.0349	18.1384	18.0661	18.0502
90	0.500	40	6	14.9156	14.9554	14.9333	14.8870
95	0.500	40	6	12.2278	12.2451	12.2362	12.1948
100	0.500	40	6	9.9428	9.9430	9.9449	9.9259
105	0.500	40	6	8.0241	8.0170	8.0222	8.0276
110	0.500	40	6	6.4315	6.4249	6.4271	6.4504
115	0.500	40	6	5.1230	5.1160	5.1174	5.1498
120	0.500	40	6	4.0581	4.0505	4.0521	4.0863
100	0.500	40	2	10.7719	10.7703	10.7667	10.7862
100	0.500	40	4	10.3421	10.3386	10.3384	10.3466
100	0.500	40	6	9.9428	9.9430	9.9449	9.9259
100	0.500	40	8	9.5686	9.5774	9.5797	9.5279
100	0.500	40	10	9.2166	9.2320	9.2389	9.1533
100	0.500	30	6	7.2094	7.2140	7.2170	7.1807
100	0.500	35	6	8.5756	8.5780	8.5802	8.5530
100	0.500	40	6	9.9428	9.9430	9.9449	9.9259
100	0.500	45	6	11.3093	11.3075	11.3094	11.2977
100	0.500	50	6	12.6742	12.6705	12.6724	12.6673
100	0.083	40	6	4.3850	4.3841	4.3828	4.3912
100	0.167	40	6	6.0672	6.0655	6.0644	6.0735
100	0.250	40	6	7.3069	7.3040	7.3045	7.3100
100	0.333	40	6	8.3177	8.3129	8.3162	8.3154
100	0.417	40	6	9.1821	9.1796	9.1822	9.1730
100	0.500	40	6	9.9428	9.9430	9.9449	9.9259
100	0.583	40	6	10.6248	10.6282	10.6294	10.5996
100	0.667	40	6	11.2448	11.2516	11.2521	11.2107
100	0.750	40	6	11.8140	11.8245	11.8246	11.7710
100	0.833	40	6	12.3409	12.3551	12.3550	12.2887
100	0.917	40	6	12.8318	12.8498	12.8496	12.7704
Sec's per Put				0.00429	0.00956	0.01886	0.00619
Mean Rel. Err.				-0.0003	0.0003	0.0000	-0.0012
Root Mean Sqr'd Err.				0.0030	0.0440	0.0120	0.0346
Mean Abs. Rel. Err.				0.0003	0.0012	0.0008	0.0026
Max Rel. Err.				-0.0007	0.0100	-0.0022	-0.0072

Table 2e: Short Term Put Values, Strike Price = $100, No Dividend

Stock Price in $	Time to Exp'n	Vol-ati-lity	Risk free Rate	Std. Bin'l $N = 300$	Acc'd Bin'l $N = 300$	BBS Bin'l $N = 300$	Tri-nomial $N = 300$
80	0.500	40	6	21.6062	21.6204	21.6071	21.6056
85	0.500	40	6	18.0333	18.0088	18.0390	18.0369
90	0.500	40	6	14.9219	14.8888	14.9206	14.9199
95	0.500	40	6	12.2383	12.2232	12.2336	12.2337
100	0.500	40	6	9.9400	9.9178	9.9489	9.9439
105	0.500	40	6	8.0357	8.0333	8.0300	8.0305
110	0.500	40	6	6.4326	6.4283	6.4370	6.4363
115	0.500	40	6	5.1298	5.1296	5.1279	5.1237
120	0.500	40	6	4.0669	4.0685	4.0621	4.0599
100	0.500	40	2	10.7667	10.7665	10.7776	10.7723
100	0.500	40	4	10.3385	10.3328	10.3483	10.3432
100	0.500	40	6	9.9400	9.9178	9.9489	9.9439
100	0.500	40	8	9.5664	9.5361	9.5745	9.5696
100	0.500	40	10	9.2147	9.1834	9.2220	9.2173
100	0.500	30	6	7.2078	7.1900	7.2139	7.2104
100	0.500	35	6	8.5734	8.5569	8.5809	8.5766
100	0.500	40	6	9.9400	9.9178	9.9489	9.9439
100	0.500	45	6	11.3059	11.2864	11.3162	11.3105
100	0.500	50	6	12.6701	12.6618	12.6817	12.6753
100	0.083	40	6	4.3830	4.3836	4.3874	4.3853
100	0.167	40	6	6.0648	6.0623	6.0708	6.0678
100	0.250	40	6	7.3044	7.3000	7.3114	7.3078
100	0.333	40	6	8.3150	8.3083	8.3228	8.3187
100	0.417	40	6	9.1794	9.1635	9.1878	9.1832
100	0.500	40	6	9.9400	9.9178	9.9489	9.9439
100	0.583	40	6	10.6220	10.5996	10.6313	10.6259
100	0.667	40	6	11.2418	11.2235	11.2516	11.2458
100	0.750	40	6	11.8110	11.7890	11.8211	11.8150
100	0.833	40	6	12.3379	12.2981	12.3483	12.3418
100	0.917	40	6	12.8287	12.7695	12.8393	12.8325
Sec's per Put				0.07613	0.03742	0.08484	0.16677
Mean Rel. Err.				-0.0003	-0.0018	0.0003	-0.0001
Root Mean Sqr'd Err.				0.0056	0.0255	0.0028	0.0019
Mean Abs. Rel. Err.				0.0006	0.0021	0.0003	0.0002
Max Rel. Err.				0.0014	-0.0052	0.0003	-0.0005

Table 3a: Long Term Put Values, Strike Price = $100

Stock Price in $	Time to Exp'n	Vol-ati-lity	Risk free Rate	Div. Yld %	Average Bin'l, $N = 1,000$	Method of Lines 3 pts	Quad-ratic Formula	Broadie Detemple LBA
80	3.0	40	6	2	29.2601	29.2323	29.4377	29.2105
85	3.0	40	6	2	26.9216	26.8923	27.1423	26.8793
90	3.0	40	6	2	24.8023	24.7692	25.0614	24.7669
95	3.0	40	6	2	22.8797	22.8388	23.1704	22.8488
100	3.0	40	6	2	21.1294	21.0835	21.4484	21.1039
105	3.0	40	6	2	19.5376	19.4899	19.8774	19.5142
110	3.0	40	6	2	18.0849	18.0369	18.4418	18.0635
115	3.0	40	6	2	16.7574	16.7070	17.1280	16.7380
120	3.0	40	6	2	15.5428	15.4873	15.9239	15.5252
100	3.0	40	2	2	25.8881	25.8789	26.0215	25.9096
100	3.0	40	4	2	23.2991	23.2504	23.5325	23.2812
100	3.0	40	6	2	21.1294	21.0835	21.4484	21.1039
100	3.0	40	8	2	19.2707	19.2403	19.6423	19.2564
100	3.0	40	10	2	17.6589	17.6467	18.0490	17.6637
100	3.0	40	6	0	19.8545	19.8145	20.1088	19.8362
100	3.0	40	6	2	21.1294	21.0835	21.4484	21.1039
100	3.0	40	6	4	22.4873	22.4419	22.8834	22.4635
100	3.0	30	6	2	15.1687	15.1375	15.3873	15.1444
100	3.0	35	6	2	18.1587	18.1197	18.4274	18.1332
100	3.0	40	6	2	21.1294	21.0835	21.4484	21.1039
100	3.0	45	6	2	24.0708	24.0189	24.4408	24.0464
100	3.0	50	6	2	26.9750	26.9180	27.3966	26.9525
100	0.5	40	6	2	10.2741	10.2759	10.2728	10.2697
100	1.0	40	6	2	13.8774	13.8670	13.9142	13.8679
100	1.5	40	6	2	16.3682	16.3469	16.4627	16.3545
100	2.0	40	6	2	18.2840	18.2533	18.4476	18.2664
100	2.5	40	6	2	19.8349	19.7960	20.0743	19.8134
100	3.0	40	6	2	21.1294	21.0835	21.4484	21.1039
100	3.5	40	6	2	22.2326	22.1804	22.6327	22.2029
100	4.0	40	6	2	23.1868	23.1289	23.6681	23.1526
100	4.5	40	6	2	24.0216	23.9585	24.5829	23.9826
100	5.0	40	6	2	24.7584	24.6906	25.3979	24.7145
100	5.5	40	6	2	25.4136	25.3506	26.1288	25.3645
Sec's per Put					1.74467	0.00041	0.00070	0.00161
Mean Rel. Err.					0.0000	-0.0019	0.0148	-0.0011
Root Mean Sqr'd Err.					0.0000	0.0435	0.3485	0.0273
Mean Abs. Rel. Err.					0.0000	0.0019	0.0148	0.0012
Max Rel. Err.					0.0000	-0.0036	0.0281	-0.0019

Table 3b: Long Term Put Values, Strike Price = $100

Stock Price in $	Time to Exp'n	Vol-ati-lity	Risk free Rate	Div. Yld %	Broadie Detemple LUBA	IFD $N = 200$ $M = 200$	Method of Lines 6 points	Ho et. al. 2pt.GJ
80	3.0	40	6	2	29.2540	29.0584	29.2511	31.2905
85	3.0	40	6	2	26.9157	26.6613	26.9110	28.6945
90	3.0	40	6	2	24.7989	24.4744	24.7914	26.3315
95	3.0	40	6	2	22.8778	22.4723	22.8675	24.1829
100	3.0	40	6	2	21.1306	20.6330	21.1180	22.2304
105	3.0	40	6	2	19.5387	18.9372	19.5245	20.4563
110	3.0	40	6	2	18.0860	17.3683	18.0709	18.8438
115	3.0	40	6	2	16.7585	15.9113	16.7429	17.3777
120	3.0	40	6	2	15.5437	14.5535	15.5281	16.0436
100	3.0	40	2	2	25.8864	25.3079	25.8783	26.1582
100	3.0	40	4	2	23.3016	22.7597	23.2855	23.9900
100	3.0	40	6	2	21.1306	20.6330	21.1180	22.2304
100	3.0	40	8	2	19.2670	18.8169	19.2621	20.7281
100	3.0	40	10	2	17.6510	17.2466	17.6525	19.4074
100	3.0	40	6	0	19.8570	19.3754	19.8440	20.1638
100	3.0	40	6	2	21.1306	20.6330	21.1180	22.2304
100	3.0	40	6	4	22.4884	21.9760	22.4760	24.1792
100	3.0	30	6	2	15.1693	15.0992	15.1606	16.4188
100	3.0	35	6	2	18.1600	17.9575	18.1486	19.3269
100	3.0	40	6	2	21.1306	20.6330	21.1180	22.2304
100	3.0	45	6	2	24.0713	23.1398	24.0579	25.1172
100	3.0	50	6	2	26.9741	25.4077	26.9609	27.9775
100	0.5	40	6	2	10.2750	10.2614	10.2671	10.3211
100	1.0	40	6	2	13.8796	13.8578	13.8678	14.0636
100	1.5	40	6	2	16.3712	16.3158	16.3574	16.7424
100	2.0	40	6	2	18.2869	18.1500	18.2724	18.8788
100	2.5	40	6	2	19.8371	19.5442	19.8234	20.6739
100	3.0	40	6	2	21.1306	20.6330	21.1180	22.2304
100	3.5	40	6	2	22.2326	21.4880	22.2215	23.6089
100	4.0	40	6	2	23.1857	22.1943	23.1761	24.8487
100	4.5	40	6	2	24.0197	22.7027	24.0114	25.9769
100	5.0	40	6	2	24.7564	23.1330	24.7489	27.0131
100	5.5	40	6	2	25.4121	23.4786	25.4048	27.9721
Sec's per Put					0.00572	0.16261	0.00196	0.00127
Mean Rel. Err.					-0.0000	-0.0273	-0.0005	0.0524
Root Mean Sqr'd Err.					0.0027	0.7551	0.0112	1.2849
Mean Abs. Rel. Err.					0.0001	0.0273	0.0005	0.0524
Max Rel. Err.					-0.0004	-0.0761	-0.0009	0.1007

Table 3c: Long Term Put Values, Strike Price = $100

Stock Price in $	Time to Exp'n	Vol-ati-lity	Risk free Rate	Div. Yld %	Geske Johnson 2 points	Chesney Exp'l Formula	Method of Lines 9 points
80	3.0	40	6	2	31.0305	28.5900	29.2649
85	3.0	40	6	2	28.4794	26.2560	26.9247
90	3.0	40	6	2	26.1543	24.1515	24.8048
95	3.0	40	6	2	24.0375	22.2477	22.8807
100	3.0	40	6	2	22.1114	20.5272	21.1310
105	3.0	40	6	2	20.3591	18.9643	19.5373
110	3.0	40	6	2	18.7646	17.5435	18.0835
115	3.0	40	6	2	17.3131	16.2432	16.7554
120	3.0	40	6	2	15.9911	15.0632	15.5405
100	3.0	40	2	2	26.1543	25.5821	25.8989
100	3.0	40	4	2	23.9521	22.8125	23.2937
100	3.0	40	6	2	22.1114	20.5272	21.1310
100	3.0	40	8	2	20.4765	18.6322	19.2814
100	3.0	40	10	2	18.9743	17.0258	17.6564
100	3.0	40	6	0	20.0641	19.2356	19.8500
100	3.0	40	6	2	22.1114	20.5272	21.1310
100	3.0	40	6	4	24.0547	21.9160	22.4828
100	3.0	30	6	2	16.2614	14.6968	15.1652
100	3.0	35	6	2	19.1926	17.6211	18.1770
100	3.0	40	6	2	22.1114	20.5272	21.1310
100	3.0	45	6	2	25.0085	23.4065	24.0781
100	3.0	50	6	2	27.8760	26.2564	26.9693
100	0.5	40	6	2	10.3195	10.1600	10.2694
100	1.0	40	6	2	14.0553	13.6417	13.8750
100	1.5	40	6	2	16.7200	16.0236	16.3583
100	2.0	40	6	2	18.8338	17.8412	18.2796
100	2.5	40	6	2	20.5970	19.3062	19.8278
100	3.0	40	6	2	22.1114	20.5272	21.1310
100	3.5	40	6	2	23.4373	21.5657	22.2201
100	4.0	40	6	2	24.6135	22.4633	23.1858
100	4.5	40	6	2	25.6670	23.2484	24.0174
100	5.0	40	6	2	26.6172	23.9418	24.7546
100	5.5	40	6	2	27.4786	24.5589	25.4102
Sec's per Put					0.00127	0.02202	0.00577
Mean Rel. Err.					0.0455	-0.0272	-0.0000
Root Mean Sqr'd Err.					1.1049	0.5926	0.0060
Mean Abs. Rel. Err.					0.0455	0.0272	0.0002
Max Rel. Err.					0.0813	-0.0358	0.0010

Table 3d: Long Term Put Values, Strike Price = $100

Stock Price in $	Time to Exp'n	Vol- ati- lity	Risk free Rate	Div. Yld %	Bunch Johnson modGJ	Integral Method $N = 6$	Sub'n & Yu 3 points
80	3.0	40	6	2	29.9382	29.5987	29.7147
85	3.0	40	6	2	27.3733	27.2300	27.2386
90	3.0	40	6	2	25.1566	25.0805	25.0136
95	3.0	40	6	2	23.1413	23.1280	23.0132
100	3.0	40	6	2	21.3092	21.3522	21.2121
105	3.0	40	6	2	19.6431	19.7349	19.5870
110	3.0	40	6	2	18.1558	18.2598	18.1173
115	3.0	40	6	2	16.8100	16.9126	16.7846
120	3.0	40	6	2	15.5775	15.6807	15.5729
100	3.0	40	2	2	25.9012	25.9213	25.9831
100	3.0	40	4	2	23.3798	23.4149	23.3949
100	3.0	40	6	2	21.3092	21.3522	21.2121
100	3.0	40	8	2	19.5722	19.6108	19.3745
100	3.0	40	10	2	18.0374	18.1180	17.8117
100	3.0	40	6	0	20.0641	20.0238	19.6581
100	3.0	40	6	2	21.3092	21.3522	21.2121
100	3.0	40	6	4	22.6305	22.7249	22.7568
100	3.0	30	6	2	15.3314	15.3862	15.2220
100	3.0	35	6	2	18.3298	18.3787	18.2270
100	3.0	40	6	2	21.3092	21.3522	21.2121
100	3.0	45	6	2	24.2608	24.2967	24.1674
100	3.0	50	6	2	27.1775	27.2044	27.0850
100	0.5	40	6	2	10.2679	10.2747	10.2813
100	1.0	40	6	2	13.8904	13.9006	13.8756
100	1.5	40	6	2	16.4070	16.4277	16.3657
100	2.0	40	6	2	18.3487	18.3901	18.2948
100	2.5	40	6	2	19.9434	19.9959	19.8742
100	3.0	40	6	2	21.3092	21.3522	21.2121
100	3.5	40	6	2	22.4882	22.5226	22.3725
100	4.0	40	6	2	23.5202	23.5485	23.3967
100	4.5	40	6	2	24.4327	24.4585	24.3129
100	5.0	40	6	2	25.2456	25.2735	25.1416
100	5.5	40	6	2	25.9739	26.0091	25.8980
Sec's per Put					0.00895	0.01404	0.00523
Mean Rel. Err.					0.0095	0.0109	0.0053
Root Mean Sqr'd Err.					0.2692	0.2698	0.1853
Mean Abs. Rel. Err.					0.0095	0.0109	0.0059
Max Rel. Err.					0.0232	0.0260	0.0191

Table 3e: Long Term Put Values, Strike Price = $100

Stock Price in $	Time to Exp'n	Vol-ati-lity	Risk free Rate	Div. Yld %	Std. Bin'l $N = 300$	Acc'd Bin'l $N = 300$	BBS Bin'l $N = 300$	Tri-nomial $N = 300$
80	3.0	40	6	2	29.2609	29.3264	29.2613	29.2545
85	3.0	40	6	2	26.9133	26.8994	26.9226	26.9158
90	3.0	40	6	2	24.8105	24.7751	24.8042	24.7947
95	3.0	40	6	2	22.8876	22.8051	22.8816	22.8688
100	3.0	40	6	2	21.1185	21.0049	21.1329	21.1183
105	3.0	40	6	2	19.5493	19.4631	19.5401	19.5256
110	3.0	40	6	2	18.0851	17.9793	18.0871	18.0744
115	3.0	40	6	2	16.7645	16.6677	16.7594	16.7499
120	3.0	40	6	2	15.5485	15.4619	15.5452	15.5386
100	3.0	40	2	2	25.8700	25.8840	25.8948	25.8799
100	3.0	40	4	2	23.2858	23.1857	23.3044	23.2900
100	3.0	40	6	2	21.1185	21.0049	21.1329	21.1183
100	3.0	40	8	2	19.2610	19.2096	19.2724	19.2571
100	3.0	40	10	2	17.6498	17.7031	17.6587	17.6421
100	3.0	40	6	0	19.8445	19.7690	19.8571	19.8407
100	3.0	40	6	2	21.1185	21.0049	21.1329	21.1183
100	3.0	40	6	4	22.4748	22.4056	22.4914	22.4781
100	3.0	30	6	2	15.1612	15.0947	15.1710	15.1615
100	3.0	35	6	2	18.1495	18.0553	18.1616	18.1497
100	3.0	40	6	2	21.1185	21.0049	21.1329	21.1183
100	3.0	45	6	2	24.0581	23.9384	24.0747	24.0568
100	3.0	50	6	2	26.9606	26.8312	26.9793	26.9575
100	0.5	40	6	2	10.2677	10.2588	10.2773	10.2723
100	1.0	40	6	2	13.8694	13.8443	13.8814	13.8742
100	1.5	40	6	2	16.3592	16.2799	16.3725	16.3635
100	2.0	40	6	2	18.2743	18.2214	18.2882	18.2773
100	2.5	40	6	2	19.8245	19.7229	19.8388	19.8260
100	3.0	40	6	2	21.1185	21.0049	21.1329	21.1183
100	3.5	40	6	2	22.2211	22.1163	22.2354	22.2187
100	4.0	40	6	2	23.1749	23.0871	23.1891	23.1705
100	4.5	40	6	2	24.0090	23.9453	24.0231	24.0019
100	5.0	40	6	2	24.7455	24.6961	24.7594	24.7367
100	5.5	40	6	2	25.3999	25.3557	25.4134	25.3879
Sec's per Put					0.07485	0.03667	0.08333	0.16667
Mean Rel. Err.					-0.0004	-0.0037	0.0001	-0.0005
Root Mean Sqr'd Err.					0.0105	0.0903	0.0032	0.0121
Mean Abs. Rel. Err.					0.0005	0.0040	0.0001	0.0005
Max Rel. Err.					-0.0007	-0.0059	0.0003	-0.0010

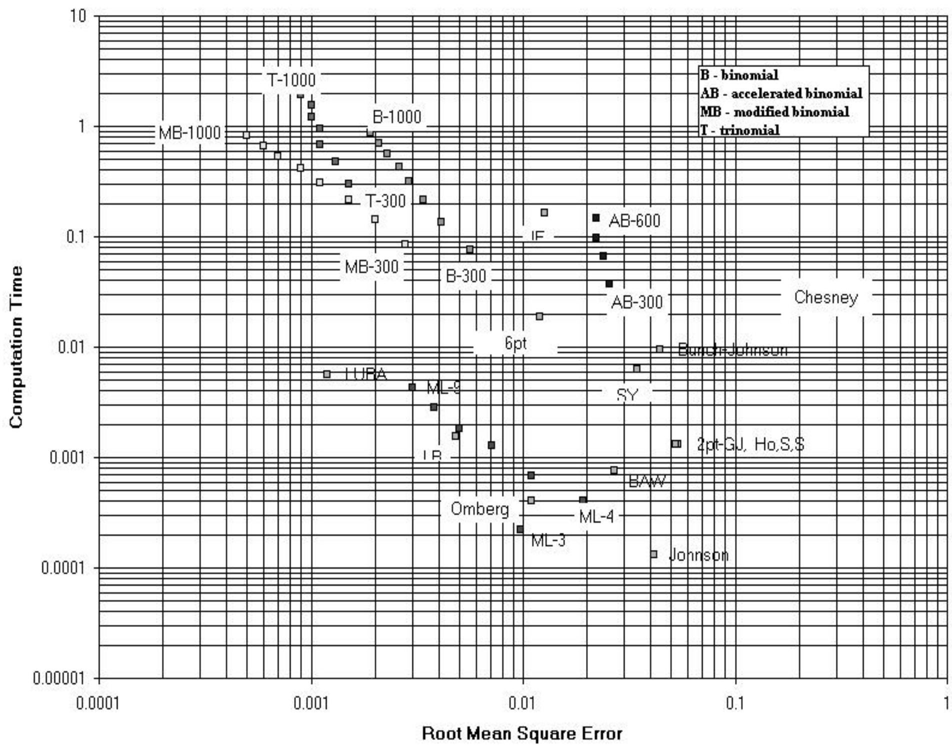

Figure 1: Short Term Options with No Dividend

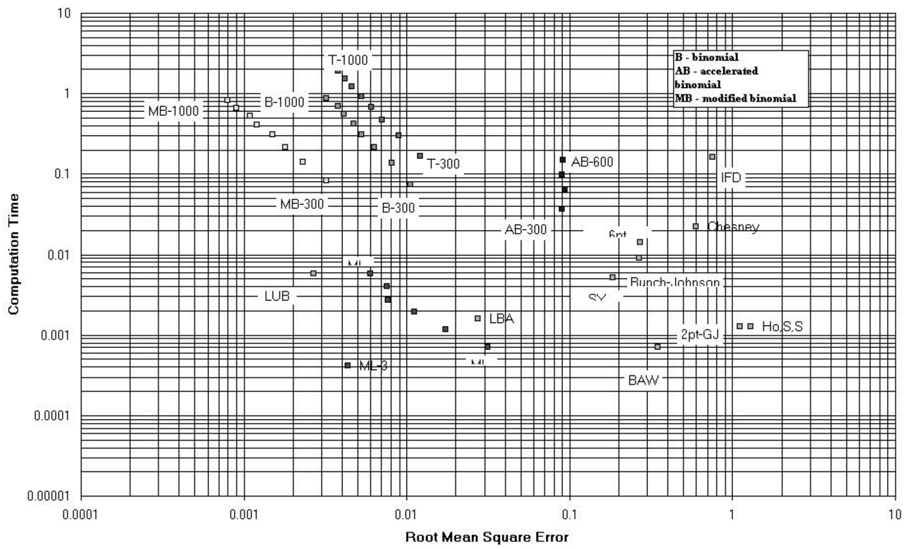

Figure 2: Long Term Options with Dividend