Hedging with Options

Peter Carr
Bloomberg LP/Courant Institute, NYU

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Co-workers

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Motivation

- The classic prescription for hedging the market risk associated with derivatives positions is to restrict the possible process dynamics sufficiently so that the payoff can be spanned by (completely) dynamic trading in the underlying asset(s). This approach introduces model risk.
- We explore alternative strategies for hedging claims written on the price path of a single underlying asset.
- When many derivatives trade on the same underlying asset's price path, there can be multiple perfect hedges under the restricted process dynamics.
- We find that for some claims, some of these alternative hedges succeed when the restrictions are lifted.

Robust Hedging

- A robust hedge strategy is one that theoretically works for a wide class of models.
- For example, PCP implies that the sale of a European call can be robustly hedged by buying the underlying stock on margin and also buying the right put.
- As the example shows, robust hedging strategies typically work only for a small class of claims being hedged. Also, they succeed even under stochastic volatility and jumps. In fact, they succeed even though volatility of volatility and the jump arrival rates are unknown. Hence, model risk is largely overcome.
- There may need to be some restrictions on the stochastic process for the underlying asset price. For example, a long forward position can be robustly replicated by buying and borrowing only by assuming that dividends are suitably restricted.
- Robust hedging strategies may or may not be (fully) dynamic.

Overview

There are 5 parts to this talk:

1. Hedging with Static Positions in European Options and their Underlying
2. Hedging with (just) Semi-Static Trading in the Underlying
3. Static Option Hedging plus Semi-Static Trading in their Underlying
4. Semi-Static Hedging in (just) European Options
5. Fully Dynamic Hedging in European Options and their Underlying

Given the time constraint, I will give just one or two examples of each.

Assumptions

- My entire talk assumes that there are no frictions, no illiquidity, no default risk, and no arbitrage opportunities.

- The assets which trade continuously without frictions, illiquidity, default, or arbitrage include:
  - bonds of all maturities
  - stocks, (not necessarily with limited liability),
  - equity forwards of all maturities and delivery prices
  - equity futures of all maturities
  - standard European options of all strikes and maturities.

- In general, future stock prices, interest rates, and dividends are arbitrarily random, unless specifically indicated otherwise.

Part 1: Static Hedging with European Options

- Appendix 1 proves that for any generalized function $f(S)$ and any scalar $\kappa \geq 0$:

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) \leftarrow \text{ tangent approximation} \\ &\quad + \int_{\kappa}^{\infty} f''(K)(S-K)^{+} dK + \int_{0}^{\kappa} f''(K)(K-S)^{+} dK \leftarrow \text{ tangent correction.} \end{aligned}$$

- This decomposition may be interpreted as a Taylor series expansion with remainder of the final payoff $f(\cdot)$ about the expansion point κ.

- The first two terms give the tangent to the payoff at κ; the last two terms continuously bend this tangent so it conforms to the nonlinear payoff.

- The payoff of an arbitrary claim has been decomposed into the payoff from $f(\kappa)$ bonds, $f'(\kappa)$ forward contracts with delivery price κ, $f''(\kappa)dK$ calls struck above κ, and $f''(\kappa)dK$ puts struck below.

From Payoffs to Prices

- Recall the decomposition of the payoff function $f(S)$:

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) \\ &\quad + \int_0^{\kappa} f''(K)(K-S)^+ dK + \int_{\kappa}^{\infty} f''(K)(S-K)^+ dK. \end{aligned}$$

- No arbitrage implies that the initial value $V_0[f]$ can be expressed in terms of the initial prices of the bond B_0, calls $C_0(K)$, and puts $P_0(K)$ respectively:

$$\begin{aligned} V_0[f] &= f(\kappa)B_0 + f'(\kappa)[C_0(\kappa) - P_0(\kappa)] \\ &\quad + \int_0^{\kappa} f''(K)P_0(K)dK + \int_{\kappa}^{\infty} f''(K)C_0(K)dK. \end{aligned}$$

- When $\kappa = F_0$, the second term vanishes by PCP, and the value decomposes as:

$$V_0[f] = \underbrace{f(F_0)B_0}_{\text{intrinsic value}} + \underbrace{\int_0^{F_0} f''(K)P_0(K)dK + \int_{F_0}^{\infty} f''(K)C_0(K)dK}_{\text{time value}}.$$

Example: In-The-Money Call

- Recall the decomposition into intrinsic and time value:

$$V_0[f] = f(F_0)B_0 + \int_0^{F_0} f''(K)P_0(K)dK + \int_{F_0}^{\infty} f''(K)C_0(K)dK.$$

- For example, suppose the final payoff is that of an in-the-money European call, i.e. $f(S) = (S - K_c)^+$, $K_c < F_0$. Formally using the above decomposition with $\kappa = F_0$ gives:

$$C_0(K_c) = (F_0 - K_c)B_0 + P_0(K_c),$$

  which is Put Call Parity.

- Thus the equation at the top is a generalization of PCP to multiple options. Can we generalize further to path-dependent payoffs?

Part 2: Semi-Static Trading in the Underlying

- For certain path-dependent payoffs, the payoff can be spanned by just semi-static trading in the underlying asset.
- By semi-static, we mean that trades can occur each time that the path must be monitored to compute the payoff of the path-dependent claim. Hence, if the path is continuously monitored as for some barrier options, trading might be continuous. We will therefore focus on (the bigger class of) path-dependent claims with discrete path monitoring.
- As usual, no assumption is made regarding the stochastic process of the underlying. This is useful because even though a model may have worked well in the past, there is no guarantee that it will continue to work well in the future.

Example: Serial Covariance Contract

- Suppose that we partition the time set $[0, T)$ into n time intervals of the form $[t_i, t_{i+1})$, where:
$$0 \equiv t_0 \leq t_1 \leq t_2 \leq ...t_{n-1} \leq t_n \equiv T.$$
- Let F_i denote the futures price at time t_i for maturity T. We assume marking-to-market occurs at each t_i.
- Suppose that the payoff on a serial covariance contract is defined as:
$$Cov_n \equiv \frac{1}{n-1} \sum_{i=1}^{n-1} \left(\frac{F_i - F_{i-1}}{F_{i-1}} \right) \left(\frac{F_{i+1} - F_i}{F_i} \right).$$
In words, the payoff is the average of the products of adjacent returns.
- How do we hedge and price this highly path-dependent payoff?

Robust Hedging of Covariance Contracts

- Recall that the payoff on a covariance contract was defined as:

$$Cov_n \equiv \frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1}-F_i}{F_i}\right).$$

- Let $r(t)$ be the deterministic spot interest rate at time t.

- Suppose we do nothing from day 0 to day 1. If we hold $\frac{e^{-\int_{t_{i+1}}^{t_n} r(u)du}(F_i - F_{i-1})}{(n-1)F_iF_{i-1}}$ futures contracts from time t_i to time $t_{i+1}, i = 1, \ldots, n-1$, then we receive $\left(\frac{e^{-\int_{t_{i+1}}^{t_n} r(u)du}(F_i - F_{i-1})}{(n-1)F_iF_{i-1}}\right) \times (F_{i+1} - F_i)$ in marking-to-market proceeds at time t_{i+1}.

- The future value of these proceeds are $\left(\frac{(F_i - F_{i-1})}{(n-1)F_iF_{i-1}}\right) \times (F_{i+1} - F_i)$ by time t_n.

- Summing over $i = 1, 2, \ldots, n-1$, the sum of the future values of the marking-to-market proceeds by time t_n is $\frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1}-F_i}{F_i}\right)$.

Pricing Covariance Contracts

- Recall that the payoff on a covariance contract was defined as:

$$Cov_n \equiv \frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right).$$

- The last page showed that by semi-statically trading futures, this payoff could be perfectly replicated.
- As the initial position is zero futures and as the futures trading strategy is trivially self-financing, the arbitrage-free value of the payoff on the covariance contract is zero.

Why?

- Recall that the payoff on a covariance contract was defined as:

$$Cov_n \equiv \frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right).$$

- It is well known that no arbitrage implies the existence of a probability measure $\mathbb{Q}$ equivalent to the original measure $\mathbb{P}$ such that the futures price is a martingale.
- This martingale is adapted to the futures price process.
- Hence, payoffs of the form $\int_0^T N_t^f dF_t$ are priced at 0 so long as N_t^f just depends on time and the futures price path up to t.
- The covariance contract payoff defined above is just a special case.
- All martingales have increments which are uncorrelated. All we have done is to demonstrate the trading strategy in futures that enforces this result.
- One can also trade cross auto-covariance (for zero if written on futures prices).

Part 3: Static Option + Semi-static Underlying

- For certain path-dependent payoffs,the payoff can be spanned by combining a static position in options with semi-static trading in the underlying.
- As usual, no assumption on the stochastic process for the underlying is needed.
- We illustrate with two examples. The first has discrete path monitoring while the second has continuous path monitoring and (multiple) early exercise.

Example 1: Local Variation

- Consider a finite set of discrete times $\{t_0, t_1, \ldots, t_n\}$ at which one can trade futures contracts.
- Let F_i denote the price traded at on day i, for $i = 0, 1, \ldots, n$.
- For any $K > 0$, consider the payoff:

  $$\sum_{i=1}^{n} [1(F_{i-1} \leq K)(F_i - K)^+ + 1(F_{i-1} > K)(K - F_i)^+],$$

  Thus for each time i, the payoff is zero if there is no cross of K. If there is a cross from below, the payoff is $F_i - K > 0$. If there is a cross from above, the payoff is $K - F_i > 0$.
- We refer to this payoff as the variation of the F process at time n, localized to the strike K.

Robust Hedging of Local Variation

- It is a tautology that the target payoff

$$\begin{aligned}&\sum_{i=1}^{n}\left[1(F_{i-1}\leq K)(F_i-K)^+ + 1(F_{i-1}>K)(K-F_i)^+\right]\\ &= (F_n-K)^+-(F_0-K)^+-\sum_{i=1}^{n}1(F_{i-1}>K)(F_i-F_{i-1}).\end{aligned}$$

- Hence, the claim paying the local variation can be hedged by buying a call and eliminating its intrinsic value whenever it is positive by shorting the forward contract with delivery price K.
- The fair price of the local variation is the initial premium of the OTM option with the same underlying, strike, and maturity.

Observations and Generalizations

- By integating K from 0 to infinity, one can also create the payoff $\sum_{i=1}^{n}(F_i - F_{i-1})^2$.
- Dividing by n, this payoff is the floating part of a price variance swap.
- More generally, one can create the payoff $\sum_{i=1}^{n} g(F_i, F_{i-1})$ by combining semi-static trading in the underlying with a static position in options of all strikes maturing at t_n if and only if $g_{11}(F_i, F_{i-1})$ is independent of F_{i-1}.
- This condition is violated for standard variance swaps and hence exact replication requires further assumptions.

Example 2: Hyper Options

- To the pantheon of exotic options, we introduce HYPER options (High Yielding Performance Enhancing Reversible options).
- As usual, a hyper option is issued as either a call or a put.
- A hyper option is similar to an American option in that it can be exercised early, but it also differs from an American option in that it can be exercised an unlimited number of times.
- Exercising a hyper option not only locks in the exercise value, but it also turns a hyper call into a hyper put and vice versa.
- Thus after a hyper call is first exercised, it can be exercised next as a put, then as a call, etc. The strike, maturity, and underlying are never changed.
- Since a hyper option can be exercised an unlimited number of times, all of the exercise proceeds are deferred without interest to maturity.
- As usual, a hyper option need never be exercised, so it has nonnegative value.

Hyper Options on Forward Prices

- In this presentation, we will only consider hyper options written on the forward price F of some underlying asset. We assume that both the hyper option and the forward contract mature at some fixed date T.
- If a hyper call is exercised at any time $t \in [0, T]$, then the owner will receive $F_t - K$ at T, where K denotes the strike price of the hyper option.
- Exercising the hyper call converts it into the corresponding hyper put.
- We do not require that the hyper call be ITM for it to be exercised. If the owner exercises his hyper call while $F < K$ to obtain the ITM hyper put, then $F - K$ is negative so the owner owes $K - F$ to the writer at maturity.
- If a hyper put is exercised at any time $t \in [0, T]$, then the owner will receive $K - F_t$ at T and the hyper put reverses into the corresponding hyper call.
- At maturity, the hyper option can be exercised for the final time or it can expire worthless.

Get Plenty of Exercise

- We restrict ourselves to exercise strategies which include exercising at maturity if and only if it is ITM.
- We refer to such a strategy as *sensible*. Sensible strategies permit exercise prior to maturity as well.
- We say that a sensible exercise strategy is *optimal* if it is value maximizing.
- Depending on the price path which is realized, we will show that it can be optimal for the owner of a hyper option to exercise early one or more times.
- In fact, at any time prior to maturity, there is always positive probability of multiple optimal early exercises.
- Thus, the writer of a hyper option must find a hedging strategy which defends against these multiple optimal early exercises.
- Ideally, this hedging strategy would also be immune to model risk.

The Hyper American

- Recall that a hyper option is a multiply exerciseable American option whose polarity switches on each exercise.
- Since hyper options can potentially be exercised infinitely often, all exercise proceeds are deferred without interest to maturity.
- When the hyper option is written on a forward price as we assume, then at any time there is positive probability of multiple optimal early exercises.
- All of this suggests that a hyper option has greater value than a standard American option on the forward price (which has a positive early exercise premium).

Objects May Appear Larger...

- Assuming only frictionless markets and no arbitrage, we show that a hyper option has exactly the same value as the corresponding European option, regardless of the model.
- Thus, no arbitrage forces the hyper call to have the same value as the European call with the same underlying, strike, and maturity. The analogous statement holds for puts.
- The reason for these surprising results is that all sensible exercise strategies are also optimal.
- Note that this result differs from Merton's classical result for American calls on non-dividend paying stocks. For these options, the optimal exercise strategy is to wait to maturity and exercise if and only if the call is ITM then.

Robust Hedging of Hyper Options

- Let C_t^h and P_t^h denote the respective prices at time $t \in [0,T]$ of hyper calls & puts with fixed strike K & fixed maturity T.
- Let C_t^e and P_t^e denote the corresponding European option prices, which satisfy:

$$C_t^e - P_t^e = B_t(F_t - K), \qquad t \in [0,T].$$

- Consider the following *polarity matching strategy* for hedging the sale of a hyper option: Buy a call, and
  1. If the owner is holding the hyper option as a call, hold nothing else, otherwise:
  2. If the owner is holding the hyper option as a put, also be short a forward contract with delivery price K. Thus, the net position is long one synthetic put.
- From put call parity written above, this strategy perfectly replicates the payoffs to a hyper option and hence we conclude from no arbitrage that:

$$C_t^h = C_t^e \qquad P_t^h = P_t^e, \qquad t \in [0,T].$$

Concluding Remarks on Hyper Options

- The two examples in this part are linked. The local variation of the price path arises if the owner of the hyper option adopts an exercise strategy which monitors the path discretely and exercises as soon as the option is ITM.
- Other exercise strategies can be used to generate upper bounds on the number of upcrosses or downcrosses of a given spatial interval.
- Roger Lee and I have also looked at hyper options on the spot price and other variations on the hyper option payoff.

Part 4: Semi-Static Hedging in European Options

- For certain path-dependent payoffs,the payoff can be spanned by just a semi-static position in standard European options.
- No position is taken in the underlying at all. Thus, the hedge is especially useful when options trade but their underlying does not.
- In contrast to the work thus far, some assumptions on the stochastic process for the underlying are needed. However, the restrictions made are generally weaker than in the literature.
- We illustrate with two examples. The first assumes no risk-neutral drift in the price and that the risk-neutral distribution for the terminal price is symmetric about the current price. The second further assumes continuity of prices over time. Both examples allow stochastic volatility where the volatility process is unknown.

Example 1: Passport Option

- Consider a fictitious dynamic trading strategy conducted over the time interval $[0,T]$ where the number of shares held is restricted to be ±1. Assuming zero rates, the running P&L at time t is defined by:

$$\pi_t \equiv \int_0^t c_s dS_s, \qquad t \in [0,T], \text{ where } c_s = \pm 1.$$

- Letting PP_t denote the arbitrage-free value of a passport option at time t, a passport option has a final payoff at T which is the positive part of the final P&L:

$$PP_T = (\pi_T - k)^+,$$

  where $k \in \mathbb{R}$ is the strike price.

- Passport options are sometimes defined by letting $c_t \in [-1,+1]$. However, it is easy to show that the value maximizing strategy restricts the range to $c_t = \pm 1$ for all $t \in [0,T]$.

Assumptions

- We assume no interest rates or dividends for simplicity. If the underlying is a forward price, then we can allow deterministic interest rates. If the underlying is a spot price, then we require that the dividend yield always equals this deterministic interest rate.

- We further assume that the risk-neutral distribution for the terminal price conditioned on the current price is always symmetric about this current price.

- For analytical convenience, we also assume that the passport option holder is contractually restricted to only change parity a countable number of times. In practice, this is always the case.

- Note that jumps in price are allowed.

Arithmetic Put Call Symmetry

- Our central assumption is that a standard put and call of the same moneyness, maturity, and underlying have the same market price. We refer to this critical condition as arithmetic put call symmetry (APCS).

- Let $C_t(K_c)$ and $P_t(K_p)$ respectively denote the market call and put prices for strikes $K_c \in \mathbb{R}$ and $K_p \in \mathbb{R}$ at time $t \in [0, T]$. Let $M_t^c \equiv S_t - K^c$ define the moneyness of a call with strike K_c and let $M_t^p \equiv K^p - S_t$ define the moneyness of a European put with strike K^p.

- Using this notation, our APCS condition is that:

  $$C_t(S_t - M_t^c) = P_t(S_t + M_t^p), \quad t \in [0, T],$$

  when $M_t^c = M_t^p$.

- For future use, we define the moneyness $m_t \in \mathbb{R}$ of a passport option as:

  $$m_t \equiv \pi_t - k.$$

Intuition on Robust Hedge

- Call and put payoffs can be written in terms of their moneyness at $t \in [0, T]$:

$$\begin{aligned} C_T(K_c) &= (S_T - K_c)^+ = \left(M_t^c + \int_t^T (+1) dS_u \right)^+ \\ P_T(K_p) &= (K_p - S_T)^+ = \left(M_t^p + \int_t^T (-1) dS_u \right)^+ . \end{aligned}$$

- The passport option payoff can also be written in terms of its moneyness at t:

$$PP_T = (m_t + \int_t^T c_u dS_u)^+.$$

- At any $t \in [0, T]$, the hedger can always find a standard option with the same parity and moneyness as the passport option. If the holder of the passport never changes parity after time t, then a static position in this option replicates the passport payoff. The APCS condition guarantees that the hedger can always *costlessly* switch into the standard option with the right parity and moneyness.

Robust Hedge of a Passport Option

- Let S_t^-, and m_t^- respectively denote the stock price, and the passport moneyness at the last switch time at or before time t. Given our assumption that switches do not occur continuously over time, S_t^-, and m_t^- will almost always be pure jump processes, which are RCLL.

- Our hedging strategy only uses European options of maturity T. Let $N_t^c(K)$ and $N_t^p(K)$ respectively denote the number of calls and puts of strike $K \in \mathbb{R}$ which are held at time $t \in [0, T]$.

- Consider the following dynamic trading strategy in calls and puts:

$$\begin{aligned} N_t^c(K) &= 1(c_t = 1)\delta(K - (S_t^- - m_t^-)) \\ N_t^p(K) &= 1(c_t = -1)\delta(K - (S_t^- + m_t^-)), \quad t \in [0, T], \end{aligned}$$

  where $\delta(\cdot)$ denotes the Dirac delta function.

Robust Hedge of a Passport Option (Con'd)

- It can be shown that this option trading strategy has the following properties:
  1. it is not anticipating.
  2. it is semi-static, in that it is static between switch times.
  3. at a non-switching time, the hedger always holds just one option whose parity matches that of the passport and whose moneyness at the last switch time matched the moneyness of the passport then. At a switch time τ_i, the option held at τ_i has parity and moneyness equal to that of the passport at τ_i.
  4. the portfolio is self-financing, replicating, and has the same initial cost as the call or put with the same initial moneyness as the passport.

- As we have identified a non-anticipating replicating self-financing trading strategy, the passport option has the same initial value as a standard call or put with the same initial moneyness:

$$PP_0 = V_0 = C_0(S_0 + k) = P_0(S_0 - k).$$

Example 2: Down-and-In Call

- Let τ_H denote the first passage time to some fixed barrier $H < S_0$.
- Letting DIC_t denote the arbitrage-free value of a down-and-in call option at time t, the final payoff at T is:

$$DIC_T = 1(\tau_H < T)(S_T - K)^+,$$

where $K \in \mathbb{R}$ is the strike price.

Assumptions

- We make all of the assumptions that were made for the passport option (no RN drift, price symmetry).
- We further assume that the spot price process is continuous over time.
- We are allowing for unknown stochastic volatility, but price symmetry requires that changes in instantaneous normal volatility be conditionally independent of changes in price.
- Our assumption is consistent with implied volatility always being a decreasing function of moneyness (at each maturity).

Robust Hedge of a Down-and-In Call

- Let $\tau \equiv \tau_H \wedge T$ be the earlier of the first passage time to H and the maturity date T.

- The basic form of our written DIC hedge is to hold a standard put when $t \in [0, \tau]$ and to hold a standard call when $t \in (\tau, T]$.

- More precisely, the standard put has the same underlying and maturity as the DIC and its strike is such that it will have the same moneyness at the barrier as the DIC. The relevant put strike is $2H - K$. The standard call has the same underlying, strike, and maturity as the DIC.

- Letting N_t^p and N_t^c denote the number of such puts and calls held at time $t \in [0, T]$, the strategy is:

$$N_t^p = 1_{t \in [0,\tau]} \qquad N_t^c = 1_{t \in [\tau, T]}.$$

Robust Hedge of a Down-And-In Call (Con'd)

- It can be shown that this option trading strategy has the following properties:
  1. it is not anticipating.
  2. it is semi-static, in that it is static before and after the first passage time τ_H.
  3. the portfolio is self-financing, replicating, and has the same initial cost as the put with the same moneyness that the call would have at the barrier.
- As we have identified a non-anticipating replicating self-financing trading strategy, the DIC has the same initial value as the put with the same moneyness that the call would have at the barrier:

$$DIC_0 = P_0(2H - K).$$

Much Ado About Skew

- APCS is just a sufficient condition for the long put to hedge the sale of a down-and in call.

- Really one only needs symmetry in the tails (i.e. below K_p and above K_c) of the RN PDF at the first passage time to the barrier.

- In other work, I have developed a family of put call symmetries based on displaced diffusion i.e. where normal vol is affine in price.

- When skew is unavoidable, one can think of the short DIC/long put as a way to bet on (positive) skew at the first passage time to the barrier. This is especially interesting in currencies where skew is stochastic.

Part 5: Dynamic Hedging in European Options

- Consider a European-style claim that pays out $h(F_t, \langle \ln F \rangle)$ at its maturity T, where h is a given function.
- Assuming that the price process is continuous over time and that the instantaneous lognormal volatility evolves independently of the price, it can be shown that such a payoff can be spanned by dynamic trading in a continuum of options of maturity T.
- For more details, see my talks on the web on trading volatility derivatives. Just google my name.

Summary

- We have shown how to replicate the payoffs from path-independent payoffs, serial covariation contracts, local variation contracts, hyper options, passport options, and barrier options.
- The last four claims all had the same price as a single Euroepan option, despite the fact that their payoff was path-dependent.
- In general, the greater the usage of options in the hedge, the less one is relying on a model.
- This reduced model risk often comes at the price of greater transactions cost. Hence one cannot escape the risk-reward tradeoff which defines modern finance.

App: Replic'g Payoffs with Bonds & Options

- For any fixed κ, the fundamental theorem of calculus implies:

$$
\begin{aligned}
f(S) &= f(\kappa) + 1_{S>\kappa} \int_{\kappa}^{S} f'(u)du - 1_{S<\kappa} \int_{S}^{\kappa} f'(u)du \\
&= f(\kappa) + 1_{S>\kappa} \int_{\kappa}^{S} \left[f'(\kappa) + \int_{\kappa}^{u} f''(v)dv \right] du \\
&\quad -1_{S<\kappa} \int_{S}^{\kappa} \left[f'(\kappa) - \int_{u}^{\kappa} f''(v)dv \right] du.
\end{aligned}
$$

- Noting that $f'(\kappa)$ is independent of u, Fubini's theorem implies:

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) + 1_{S>\kappa} \int_{\kappa}^{S} \int_{v}^{S} f''(v)\,du\,dv \\ &\quad + 1_{S<\kappa} \int_{S}^{\kappa} \int_{S}^{v} f''(v)\,du\,dv. \end{aligned}$$

- Integrating over u yields:

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) + 1_{S>\kappa} \int\limits_{\kappa}^{S} f''(v)(S-v)dv \\ &\quad + 1_{S<\kappa} \int\limits_{S}^{\kappa} f''(v)(v-S)dv \\ &= f(\kappa) + f'(\kappa)(S-\kappa) + \int\limits_{\kappa}^{\infty} f''(v)(S-v)^+ dv \\ &\quad + \int\limits_{0}^{\kappa} f''(v)(v-S)^+ dv. \end{aligned}$$

- Q.E.D. (quite easily done).

My Papers on Hedging with Options

- Here is a list of papers by myself and co-authors on the subject of hedging with options:
    1. “Frequently Asked Questions in Option Pricing Theory”, forthcoming in *Journal of Derivatives.*
    2. “Commodity Covariance Contracting”, (co-authored with T. Corso), 2001, *Energy & Power Risk Management*, **4**, 42–5.
    3. “Optimal Positioning in Derivative Securities,” (co-authored with D. Madan), 2001, *Quantitative Finance*, **1**, 1, 19–37.
    4. “Optimal Investment in Derivative Securities,” (co-authored with D. Madan and X. Jin), *Finance and Stochastics*, **5**, 1, 33-60.
    5. “Going with the Flow”, (co-authored with A. Lipton-Lifschutz and D. Madan), *Risk*, Aug. 2000, 85–89.
    6. “Static Hedging of Timing Risk,” April 1999, (co-authored with J. Picron), *Journal of Derivatives*, 57–66.

7. "Towards a Theory of Volatility Trading", (co-authored with D. Madan), Volatility, Risk Publications, R. Jarrow, ed., 417–427. Reprinted in Advances in Mathemati
Musiella, Jouini, Cvitanic, ed.
8. "Breaking Barriers", (co-authored with A. Chou), *Risk*, Sept. 1997, 139–145. Reprinted in Hedging with Trees, Risk Publications, R. Jarrow ed., 31–38.
9. "Static Simplicity", (co-authored with J. Bowie) **Risk**, August 1994, 44-50. Reprinted in Over the Rainbow, Risk Publications, R. Jarrow, ed.
10. "Alternative Characterizations of American Put Options," (co-authored with R. Jarrow and R. Myneni), *Mathematical Finance*, April 1992, 87–105.
11. "Hedging Complex Barrier Options," (co-authored with A. Chou), Apr. 1997.
12. "American Put Call Symmetry," (co-authored with M. Chesney), Nov. 1996.

- These papers can be downloaded from:
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`www.math.nyu.edu\research\carrp\papers`

References on Static Hedging of Exotics

References

[1] Acworth P., “Pricing and Hedging Barrier and Forward Start Options using Static Replication”. ING Barings working paper.

[2] Allen S., and O. Padovani, “Risk Management Using Quasi- Static Hedging,” Courant Institute working paper.

[3] Andersen L. and J. Andreasen, 2000, “Static Barriers,” *Risk*, September, 120-2.

[4] Andersen L., J. Andreasen and D. Eliezer, 2000, “Static Replication of Barrier Options: some general results,” General Re Financial Products, at www.ssrn.com

[5] Andreasen J., 2001, “Behind the Mirror” Risk Magazine forthcoming.

[6] Aparicio S. and L. Clewlow, “A Comparison of Alternative Methods for Hedging Exotics Options,” FORC working paper.

[7] Bhandari A., “Static Replication: A Genetic Algorithms Approach”.

[8] Bhandari A., "Mathematics of Derivative Securities".

[9] Bowie, J. and P. Carr, "Static Simplicity", *Risk*, August 1994, 44-50. Reprinted in *Over the Rainbow*, Risk Publications. R. Jarrow, ed.

[10] Brown H., Hobson D., and C. Rogers, 2001, "Robust Hedging of Barrier Options," *Mathematical Finance*, **11**, 3, 285–314.

[11] Davis M., W. Schachermayer, and R. Tompkins, 2000, "Pricing, No-arbitrage Bounds and Robust Hedging of Installment Options," Technische Universitat working paper, Vienna Austria.

[12] Derman, E., D. Ergener, and I. Kani, 1995, "Forever Hedged," *Risk*, *Sept.*, **7** 9, 139–145.

[13] Derman, E., D. Ergener, and I. Kani, 1995, "Static Options Replication," *Journal of Derivatives*, Summer, **2**, 4, 78–95.

[14] El Karoui, and M. JeanBlanc-Picque, 1997, "Exotic Options without Mathematics," , Jussieu working paper.

[15] Haug, E., 2001, "First-then-barrier-options", Wilmott Magazine, July.

[16] Haug, E., 1999, "Barrier Put-Call transformations", Paloma Partners working paper.

[17] Haug, E., 2000, "Closed Form Valuation of American Barrier Options," Paloma Partners working paper.

[18] Herzfeld E. and H. Konishi, 1997, Static Replication of Interest Rate Contingent Claims," MIT master's thesis.

[19] Hobson D., 1998, "Robust Hedging of the Lookback Option," *Finance and Stochastics*, **2**, 329–347.

[20] Jaeckel P. and R. Rebonato, 1999, "An Efficient and General Method to Value American-style and Equity and FX options in the presence of user-defined smiles and time-dependent volatility," NatWest working paper.

[21] Peccati, G., 2001, "A time-space hedging theory," Université Paris VI working paper

[22] Sbuelz A., 1998, "A General Treatment of Barrier Options," , London Business School working paper.

[23] Thomas B., 1998, "Exotic Options II", chapter 4 of *Handbook of Risk Management*. Alexander C., ed.

[24] Thomsen H., 1998, "Barrier Options: Evaluation and Hedging," University of Aarhus dissertation

[25] Toft K. and C. Xuan., 1998, "How Well Can Barrier Options be Hedged by a Static Portfolio of Standard Options?," *The Journal of Financial Engineering*, **7**, 2, 147–175.

[26] Tompkins R., 1999, "Power options: hedging nonlinear risks," *The Journal of Risk*, **2**, 2, 29–45.

[27] Tompkins R., 1997, “Static versus Dynamic Hedging of Exotic Options: An Evaluation of Hedge Performance via Simulation,” *Net Exposure*, **2**, November, 1–36, www.netexposure.co.uk.