

Static Hedging of Timing Risk

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Abstract

Many exotic options involve a payoff which occurs at the first passage time to a constant barrier. Although the amount to be paid is known, the time at which it is paid is not. This paper shows how a static position in vanilla European options can be used to hedge against this timing risk. We also show how the results can be used to price any barrier options.

Static Hedging of Timing Risk

1 Introduction

It is a widely held belief that the mathematics needed to price exotic options is more onerous than the corresponding mathematics for pricing vanilla options. The objective of this paper is to show that certain kinds of exotic options are more easily priced in the Black Scholes model, so long as there is a well-developed market in vanilla options. Our approach is to exploit a linearity principle, which is explicated mathematically in the next section. In words, this principle states that if a security's future value is known to be linear in some other asset prices, then in a frictionless market, the absence of arbitrage requires that the security's value at any prior time be given by this same linear relationship. We will show that this principle underlies all arbitrage pricing theory, including dynamic trading, which can be used to value securities with non-linear payoffs.

To illustrate the principle, we focus on the static hedging of barrier options, which are the most popular form of the second generation exotic options. Barrier options are otherwise vanilla options which are knocked in or out at the first time the underlying asset hits a constant barrier. For example, an up-and-in put is an otherwise vanilla put which is knocked in at the first hitting time of an upper barrier. At the hitting time, the option turns into vanilla put with a pre-specified strike and maturity. If the barrier is not hit before this maturity, the up-and-in put expires worthless. In contrast, an up-and-out put expires as a vanilla put unless the option has been knocked out previously.

When an option knocks out, a fixed rebate is often paid out as partial compensation. These rebates may trade separately from the barrier option in which they are embedded. If the rebate is to be paid at an upper (lower) barrier, the claim is termed an American binary call (put). Since the first hitting time of the barrier is not known in advance, these American binaries are subject to a kind of timing risk. The focus of this paper is primarily on the static hedging of this timing risk. We will show that the ability to value and hedge American binaries permits the corresponding valuation and hedging of *any* single barrier option.

There is a growing literature on the static hedging of barrier options. Using the assumptions of the Black[1] model, Bowie and Carr[3] first showed how to statically hedge the sale of single barrier options and lookback options. Subsequent papers extended these results to a nonzero underlying carry rate[5, 6, 11], non-lognormal distributions[7], and to multiple barrier options[6, 7]. These papers all use diffusion models for the evolution of the underlying asset, and generate analytic solutions for the values and hedges of the barrier option. In general, the hedging strategy involves static positions in a continuum of options of all strikes, with a common maturity matching that of the exotic option. In contrast, Derman, Ergener, and Kani[9, 10] pioneered a second approach which uses

a discrete time model to generate numerical solutions for the values and hedges of barrier options. The hedging strategy uses a series of options maturing with and before the barrier option. The options maturing strictly before the barrier option have a common strike equal to the barrier.

This paper combines elements of both approaches. In common with the first approach, the Black Scholes model will be used, allowing explicit analytic solutions for the values and hedges. In common with the second approach, options maturing with and before the barrier option are used as part of the hedging strategy. In contrast with both approaches, options maturing before the barrier option and struck at all levels beyond the barrier are also needed to implement the hedge. Given that a continuum of strikes and maturities are not available in practice, the fact that our approach uses a larger set of strikes and maturities suggests the procedure might work better in practice.

The structure of this paper is as follows. The next section presents the linearity principle and shows how it can be used to conduct static hedging when the payoff time is known. The following section extends the application of the linearity principle to random payoff times. In particular, we show how to value and hedge American binary calls in a simplified setting, where there are no carrying costs for the option or its underlying asset. The next section extends the analysis to nonzero carrying costs on both assets. The penultimate section illustrates how to extend our results from American binary calls to any single barrier option. The final section concludes.

2 The Linearity Principle

2.1 Mathematical Formulation

The linearity principle is used to relate the value of a security to the contemporaneous spot prices of a set of other assets. Let V_t denote the value to be determined at some time $t > 0$, and let S_{kt} denote the spot price of asset k at time t , $k = 0, 1, \dots, n$, where S_{0t} is usually taken to be the spot price of the riskless asset. Then the **Linearity Principle** can be stated as follows:

If:

$$V_\tau = \sum_{k=0}^n a_k S_{k\tau}, \quad (1)$$

for any stopping time¹ τ , then in a frictionless market, the absence of arbitrage

¹Loosely speaking, a stopping time is the random time at which a specified event occurs. An example of a stopping time is the first time this year that a pre-specified barrier is touched. Any fixed time is also considered to be a stopping time. An example of a random time which is not a stopping time is the *last* time this year that the barrier is touched.

requires:

$$V_t = \sum_{k=0}^n a_k S_{kt}, \quad (2)$$

at any prior time $t \leq \tau$ for which the constants a_i are known.

Although (1) might be viewed as being overly restrictive, we will show that this condition holds for any derivative security, when the assets S_{kt} obey diffusion processes and the times τ and t are arbitrarily close. Henceforth, we refer to (1) as the *linearity condition* and to (2) as the *linearity result*.

The proof of the linearity principle is straightforward. If the linearity result is violated, then V_t is either priced above or below the right side of (2). If it is priced above, then the security should be sold and the investor should buy a_i units of each of the n assets. If it is priced below, then the security should be bought and the a_i units of each of the n assets should be sold. In either case, the investor collects the difference at t between the left and right sides of the linearity result (2), which is positive by assumption. At τ , the linearity condition (1) implies that the whole portfolio can be liquidated at no cost.

Note that the number of assets n used in the replication may be a countable or uncountable infinity. In the latter case, we must use the continuous asset version of the Linearity Principle, which involves a continuum of spot asset prices $S_t(K)$, where K is a continuous indexing variable:

If:

$$V_\tau = \int a(K) S_\tau(K) dK, \quad (3)$$

for any stopping time τ , then in a frictionless market, the absence of arbitrage requires:

$$V_t = \int a(K) S_t(K) dK, \quad (4)$$

at any prior time $t \leq \tau$ for which the function $a(K)$ is known.

Note that the stopping times τ in (1) and (3) may or may not be random. The next subsection considers non-random stopping times, while the following section considers random stopping times.

2.2 Static Hedging of Path-Independent Securities

Assuming only that markets are frictionless and that the payoff is path-independent and occurring at a fixed time T , Breeden and Litzenberger[4] showed that any such payoff can be achieved by a portfolio of European calls and puts maturing at T . In particular, using a proof given in Carr and Madan[8], Appendix 1 shows that any twice differentiable payoff $f(S)$ can be written as:

$$f(S_T) = f(\kappa) + f'(\kappa)(S_T - \kappa) + \int_0^\kappa f''(K)(K - S_T)^+ dK + \int_\kappa^\infty f''(K)(S_T - K)^+ dK, \quad (5)$$

where κ can be any fixed constant. Thus, any such payoff can be uniquely expanded² into the payoff from a static position in $f(\kappa)$ unit discount bonds, $f'(\kappa)$ forward contracts³ with delivery price κ , and the continuum of puts struck below κ and calls⁴ struck above κ . The first two terms in (5) combine to give the tangent to the curve f at the point κ . Figure 1 illustrates a tangency at $\kappa = 1$ for the quadratic payoff $f(S) = S^2$. Since $f''(K) = 2$, the quadratic payoff is achieved by adding $2dK$ puts for all strikes K below 1 and $2dK$ calls for all strikes above 1.

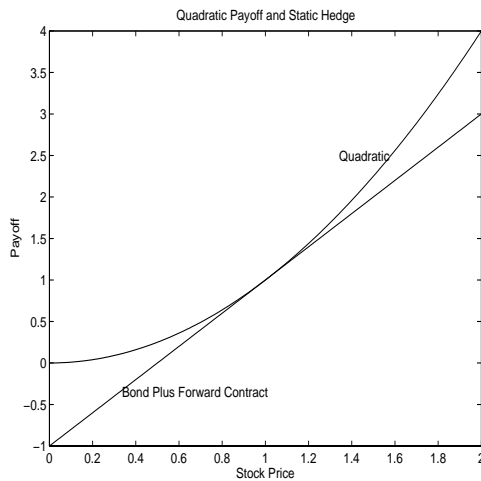


Figure 1: Figure 1: The quadratic payoff $f(S) = S^2$ is created by buying one bond, 2 forward contracts with delivery price 1, and $2dK$ puts at all strikes $K \leq 1$ and $2dK$ calls of all strikes $K \geq 1$.

Note that (5) indicates that the payoff $f(\cdot)$ is linear in the payoffs of the bond, the forward contract, and the continuum of options. Thus, the discrete and continuous versions of the Linearity Principle can be combined to determine

²Equation (5) is in fact a first order Taylor series expansion about κ with a second order remainder term.

³Note that since bonds and forward contracts can themselves be created out of options, the spectrum of options is sufficiently rich so as to allow the creation of any sufficiently smooth payoff, as shown in Breeden and Litzenberger[4].

⁴By setting κ to zero, the payoff can be generated without puts provided $f(0)$ and $f'(0)$ are both finite. Similarly, by setting κ to infinity, the payoff can be generated without calls provided $\lim_{\kappa \uparrow \infty} f(\kappa)$ and $\lim_{\kappa \uparrow \infty} f'(\kappa)$ are both finite.

the following relationship for the value of the payoff at any prior time $t \leq T$:

$$V_t = f(\kappa)B_t + f'(\kappa)I_t(\kappa) + \int_0^\kappa f''(K)P_t(K)dK + \int_\kappa^\infty f''(K)C_t(K)dK, t \in [0, T], \quad (6)$$

where B_t is the time t value of the unit bond, $I_t(\kappa)$ is the time t value of the forward contract with delivery price κ , and $P_t(K)$, $K \leq \kappa$ and $C_t(K)$, $K \geq \kappa$ are the time t values of puts and calls struck at K respectively.

An interesting special case arises if the arbitrary constant κ is taken to be the time t forward price F_t . Since the forward price changes over time, the replicating portfolio will no longer be static. However, the advantage of this formulation is that the forward contracts used in the hedge are costless ($I_t(F_t) = 0$). As a result, (6) decomposes the value of a claim with payoff $f(\cdot)$ into its intrinsic value, $f(F_t)B_t$, and its time value:

$$V_t = f(F_t)B_t + \int_0^{F_t} f''(K)P_t(K)dK + \int_{F_t}^\infty f''(K)C_t(K)dK, t \in [0, T]. \quad (7)$$

Furthermore, since the options used in the hedge are all out-of-the money-forward, the time value of the claim is expressed in terms of the time value of the options. Thus, if the payoff f is globally convex, then the time value is nonnegative. If the payoff is linear, then the claim has no time value. As a simple illustration of this result, taking $f(\cdot)$ to be the identity map:

$$f(S_T) = S_T,$$

yields the familiar cost-of-carry relation:

$$S_t = F_t B_t, \quad t \in [0, T] \quad (8)$$

when the underlying asset is assumed to pay no dividends between t and T .

In the applications which follow, the payoff $f(\cdot)$ will not be smooth. Fortunately, (5) still holds in this case if $f(\cdot)$ is interpreted as a generalized function⁵. To illustrate, suppose κ is again taken to be the time t forward price F_t , but now $f(\cdot)$ is taken to be the payoff from a call struck at $K_c < F$:

$$f(S_T) = \max[0, S_T - K_c], \quad K_c < F. \quad (9)$$

Then the delta of the payoff at maturity is a Heaviside step function:

$$f'(S_T) = \mathcal{H}(S_T - K_c), \quad K_c < F, \quad (10)$$

while the gamma at maturity is a Dirac delta function:

$$f''(S_T) = \delta(S_T - K_c), \quad K_c < F. \quad (11)$$

⁵Generalized functions always have derivatives of all orders, which are themselves generalized functions. For an introduction to generalized functions, see [14].

After this excursion into the realm of generalized functions, it is perhaps comforting to learn that the substitution of (9) to (11) into (7) yields the familiar put-call-parity:

$$C_t(K_c) = (F_t - K_c)B_t + P_t(K_c), \quad t \in [0, T]. \quad (12)$$

Thus, the cost-of-carry relation (8) and put-call-parity (12) are really just special cases of the more fundamental decomposition into intrinsic and time value (7). This equation is in turn a consequence of the linearity principle applied to non-random stopping times. The next section shows how this principle may also be applied to random stopping times.

3 Hedging American Binary Options Under No Carrying Costs

Recall that an American binary call pays one dollar at the first passage time of the underlying price to a constant upper barrier. Similarly, an American binary put pays one dollar at the first visit to a lower barrier. Both binaries are subject to timing risk, since the first hitting time of the barrier is random. The next two sections show how the linearity principle can be applied with this stopping time, so that American binary options can be statically hedged and valued. We focus on American binary calls, leaving the corresponding results for puts as an exercise for the reader. For pedagogical reasons, this section assumes that options and their underlying have no carrying cost. The next section relaxes this assumption to the case where both carrying costs are constant.

Initially, we assume that markets are frictionless and that the price process is continuous. We also assume that the probability is one that the stock price eventually hits either the barrier or the origin. At this point, we do not require that the price process be lognormal, nor do we require that vanilla options trade. However, we do assume zero carry, which in an equity context implies that interest rates and dividends are zero. This implies that the price of the riskless asset is always one dollar. Hence, the linearity principle simplifies in the current context:

If:

$$V_\tau = a + bS_\tau, \quad (13)$$

for any stopping time τ , then in frictionless markets, the absence of arbitrage requires:

$$V_t = a + bS_t, \quad (14)$$

at any prior time $t \leq \tau$ for which the constants a and b are known.

Note that since a and b are constant, the security has no time derivative.

We next show how this restricted version of the linearity principle can be applied to the static hedging of perpetual American binaries. The following subsection extends our results to finite-lived American binaries.

3.1 Perpetual American Binary Calls

By definition, a perpetual American binary call pays one dollar at the first passage time of the underlying to a constant barrier H set above the initial spot price. If the stock price hits the origin before hitting H , we assume that it is absorbed there. As a result, the perpetual American binary call becomes worthless if the origin is hit first. This American binary call is actually a special case of a more general perpetual claim, which is issued with two barriers bracketing the initial stock price. This perpetual double barrier option pays R_h dollars at the hitting time if the higher barrier H is hit first, and pays R_ℓ dollars at the hitting time if the lower barrier $L \geq 0$ is hit first. Clearly, setting $R_h = 1$, $R_\ell = 0$ and $L = 0$ yields an American binary call.

Somewhat surprisingly, there is a stopping time τ for which the payoff of the perpetual double barrier option is linear in the stock and bond. Suppose we set τ to be the earlier of the first passage time to the higher barrier, τ_h , and the first passage time to the lower barrier L , τ_ℓ . Then applying (13) along paths for which $\tau = \tau_h$ gives:

$$R_h = V_{\tau_h} = a + bH. \quad (15)$$

Along paths for which $\tau = \tau_\ell$, we have:

$$R_\ell = V_{\tau_\ell} = a + bL. \quad (16)$$

Subtracting equations gives the share position:

$$b^* = \frac{R_h - R_\ell}{H - L}. \quad (17)$$

Substituting into (16) gives the bond position:

$$a^* = R_\ell - b^*L. \quad (18)$$

Since we have assumed that the stock price must eventually hit one of the two barriers, the probability is one that the payoff of the perpetual double barrier option at τ is linear in the stock and bond:

$$PDBO_\tau = a^* + b^*S_\tau,$$

where a^* and b^* are known at any prior $t \leq \tau$. The linearity principle (14) implies that the time t price of a perpetual double barrier option is given by:

$$PDBO_t = a^* + b^*S_t, \quad S_t \in (L, H), t \leq \tau. \quad (19)$$

Setting $R_h = 1$, $R_\ell = 0$ and $L = 0$ in (17) and (18) implies $b^* = \frac{1}{H}$ and $a^* = 0$, so that the time t price of an American binary call is given by:

$$PABC_t = \frac{S_t}{H}, \quad S_t \in (0, H), t \leq \tau. \quad (20)$$

Thus, the payoffs of an American binary call can be replicated by buying the quantity $\frac{1}{H}$ shares. If the origin is hit first, both the American binary call and the shares are worthless. If the barrier H is hit first, the shares can be sold for a dollar. The value of the perpetual American binary call is graphed in Figure 2. For future use, we note from the graph and (20) that the American binary call has no time decay.

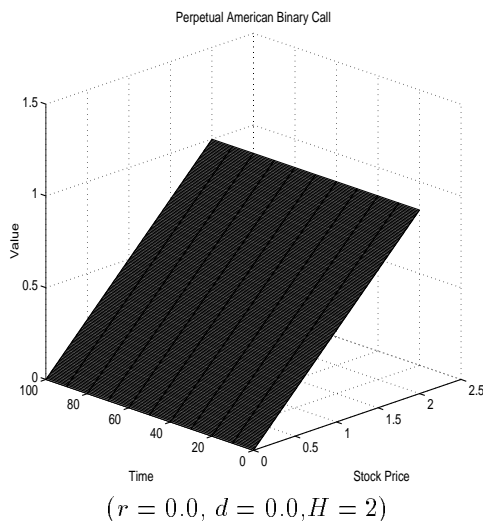


Figure 2: Figure 2: The value of a perpetual American binary call is graphed against the stock price and time.

3.2 Finite-Lived American Binary Calls

By definition, a finite-lived American binary call pays one dollar at the first passage time to a constant upper barrier, so long as this hitting time occurs before a fixed time T . If the barrier has not been hit between issuance and maturity, the option expires worthless. Henceforth, all options are assumed to be finite-lived unless specifically indicated as perpetual. Clearly, an American binary call is less valuable than its perpetual counterpart. Since both options have the same payoff along paths which hit the barrier H or the origin before T , the additional value in the perpetual option arises solely from paths which avoid both barriers, and therefore finish between 0 and H at T . Along such paths, (20) implies that the value of a perpetual option at T is:

$$PABC_T = \frac{S_T}{H}, \quad S_T \in (0, H), \tau \geq T. \quad (21)$$

This value matches the payoff from $\frac{1}{H}$ units of an up-and-out share-or-nothing put struck at H , whose value at $t \in [0, T]$ is denoted $U\&OSVNP_t$. Thus, the value at T of the finite-lived option can be obtained from that of the perpetual option by subtracting off this position in up-and-out share-or-nothing puts (see Figure 3):

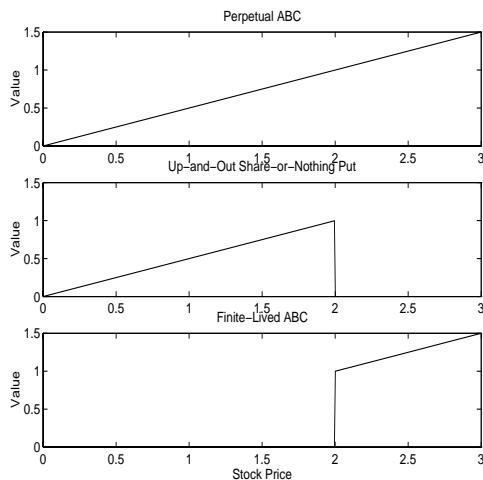


Figure 3: Figure 3: The top graph shows the value of a perpetual American binary call after one year has elapsed. The middle graph shows the value of an up-and-out share-or-nothing put at maturity. The bottom graph shows the payoff of a finite lived American binary call at maturity. It is the difference between the top graph and the middle one.

$$\begin{aligned}
 ABC_T &= PABC_T - \frac{1}{H}U\&OSVNP_T \\
 &= \frac{1}{H}(S_T - U\&OSVNP_T), \quad S_T \in (0, H), \tau \geq T, \quad (22)
 \end{aligned}$$

from (21). Like any (European) barrier option, there is an in-out parity relation for share-or-nothing puts:

$$U\&OSVNP_T = SVNP_T - U\&ISVNP_T, \quad (23)$$

where $U\&ISVNP_t$ denotes the value at $t \in [0, T]$ of an up-and-in share-or-nothing put. Substituting (23) in (22) yields:

$$ABC_T = \frac{1}{H}[S_T - SVNP_T + U\&ISVNP_T]$$

$$= \frac{1}{H}[S\mathcal{V}NC_T + U\&I\mathcal{S}\mathcal{V}NP_T], \quad S_T \in (0, H), \tau \geq T. \quad (24)$$

where $S\mathcal{V}NC_t$ denotes the value at $t \in [0, T]$ of a share-or-nothing call. Applying the linearity principle (2) to the linearity condition (24) implies that earlier values obey:

$$ABC_t = \frac{1}{H}[S\mathcal{V}NC_t - U\&O\mathcal{S}\mathcal{V}NP_t], \quad S_t \in (0, H), \tau \geq t. \quad (25)$$

We now assume that there exists a continuum of vanilla calls of all strikes struck above the barrier and maturing with the American binary call. From (5) with κ fixed at any level below H , the payoff of the share-or-nothing call can be statically replicated using a portfolio of these calls. Thus, if the up-and-in share-or-nothing put can also be statically replicated, then so can the American binary call. To accomplish this hedge, we now assume that the underlying price is a geometric Brownian motion with the constant volatility rate given by σ . This process cannot hit the origin and so τ , which was the earlier of the two hitting times, must be the hitting time of the barrier τ_h . Assuming that the underlying follows geometric Brownian motion and has no carrying cost, Carr, Ellis and Gupta[7] showed that the up-and-in share-or-nothing put can be statically replicated with H bond-or-nothing calls, with each bond-or-nothing call paying off one dollar if the stock price is above H at maturity. To see this, note that both securities expire worthless should the stock price avoid the barrier before T . If the stock price does touch the barrier, then at the hitting time τ_h , the value of the nascent share-or-nothing put is given by:

$$S\mathcal{V}NP_{\tau_h} = SN \left(\frac{\ln(H/S) - \sigma^2(T - \tau_h)/2}{\sigma\sqrt{T - \tau_h}} \right) |_{S=H} = HN(-\sigma\sqrt{T - \tau_h}/2).$$

Meanwhile, at the barrier the value of H bond-or-nothing calls is given by:

$$H\mathcal{B}\mathcal{V}NC_{\tau_h} = HN \left(\frac{\ln(S/H) - \sigma^2(T - \tau_h)/2}{\sigma\sqrt{T - \tau_h}} \right) |_{S=H} = HN(-\sigma\sqrt{T - \tau_h}/2).$$

Thus, the payoffs match in either case, and so, absence of arbitrage requires:

$$U\&I\mathcal{S}\mathcal{V}NP_t = H\mathcal{B}\mathcal{V}NC_t, \quad S_t \in (0, H), \tau_h \geq t. \quad (26)$$

Substituting this equation in (25) gives the final result:

$$ABC_t = \mathcal{B}\mathcal{V}NC_t + \frac{1}{H}S\mathcal{V}NC_t, \quad S_t \leq H, \tau_h \geq t. \quad (27)$$

Thus, in the Black Scholes model with zero interest rates and dividends, the American binary call has the same value as the sum of a bond-or-nothing call and $\frac{1}{H}$ share-or-nothing calls, with the latter two calls struck at H . If the stock

price avoids the barrier before T , then all options expire worthless. If the stock price touches the barrier before T , then the two path-independent options can be sold at the hitting time for a total of one dollar. The next section shows that a similar result holds when the options and their underlying have carrying costs.

4 Hedging American Binary Calls with Carrying Costs

For the remainder of the paper, we assume the full Black Scholes model with a constant interest rate $r > 0$ and a constant dividend yield $\delta \geq 0$. In this setting, the restricted version of the Linearity Principle becomes:

If:

$$V_\tau = a + bS_\tau, \quad (28)$$

for any stopping time τ , then in frictionless markets, the absence of arbitrage requires:

$$V_t = a\mathcal{B}_t + b\mathcal{S}_t, \quad (29)$$

at any prior time $t \leq \tau$ for which the constants a and b are known.

In (28), \mathcal{B}_t denotes the value at t of a claim that pays a dollar at the stopping time, while \mathcal{S}_t denotes the time t value of a claim that pays a share at the stopping time.

To value an American binary call, the stopping time used is the earlier of maturity and the first passage time to a constant upper barrier. However, with this definition of the stopping time, \mathcal{B}_t and \mathcal{S}_t are just the time t values of one or more American binary calls. Consequently, (29) degenerates into a useless tautology.

One way out of this problem is to consider non-random stopping times. If τ is non-random, then (29) becomes:

$$V_t = ae^{-r(\tau-t)} + bS_t e^{-\delta(\tau-t)}. \quad (30)$$

At first glance, this version of the linearity principle would seem to be of limited applicability because of the linearity requirement in (28). However, by taking τ and t to be arbitrarily close, any nonlinear relationship can be approximated arbitrarily well with a linear one, provided that the Itô calculus is applicable. We next illustrate this point by presenting a heuristic but hopefully illuminating derivation of the Black Scholes partial differential equation (p.d.e.), using only Itô's lemma and the linearity principle as described in (28) and (30). To emphasize that the claim's payoff need not be linear, we consider the pricing of a vanilla call. Let the non-random stopping time τ be the maturity date T , and let the prior time t be the instant before, i.e. $t = T - dt$. Let $C(S, t)$ denote the

value of the call given that the contemporaneous stock price is S and the time is t . By Itô's lemma:

$$dC(S_{T-dt}, T-dt) = \left[\frac{\partial}{\partial t} C(S_{T-dt}, T-dt) + \frac{\sigma^2 S_{T-dt}^2}{2} \frac{\partial^2}{\partial S^2} C(S_{T-dt}, T-dt) \right] dt + \frac{\partial}{\partial S} C(S_{T-dt}, T-dt) dS_{T-dt}. \quad (31)$$

Substituting in $dC(S_{T-dt}, T-dt) = C(S_T, T) - C(S_{T-dt}, T-dt)$ and $dS_{T-dt} = S_T - S_{T-dt}$ allows us to express the call price at maturity, $C(S_T, T)$, as a linear function of the stock price at maturity, S_T :

$$C(S_T, T) = \left\{ C(S_{T-dt}, T-dt) + \left[\frac{\partial}{\partial t} C(S_{T-dt}, T-dt) + \frac{\sigma^2 S_{T-dt}^2}{2} \frac{\partial^2}{\partial S^2} C(S_{T-dt}, T-dt) \right] dt - \frac{\partial}{\partial S} C(S_{T-dt}, T-dt) S_{T-dt} \right\} + \frac{\partial}{\partial S} C(S_{T-dt}, T-dt) S_T. \quad (32)$$

Since:

$$C(S_T, T) = \max[S_T - K, 0], \quad (33)$$

the linear relationship in (32) is clearly not holding for all S_T , but rather only for final stock prices near the current level S_{T-dt} . In other words, for diffusion processes, any nonlinear relationship is locally linear⁶. By the linearity principle (30):

$$C(S_{T-dt}, T-dt) = \left\{ C(S_{T-dt}, T-dt) + \left[\frac{\partial}{\partial t} C(S_{T-dt}, T-dt) + \frac{\sigma^2 S_{T-dt}^2}{2} \frac{\partial^2}{\partial S^2} C(S_{T-dt}, T-dt) \right] dt - \frac{\partial}{\partial S} C(S_{T-dt}, T-dt) S_{T-dt} \right\} e^{-rdt} + \frac{\partial}{\partial S} C(S_{T-dt}, T-dt) S_{T-dt} e^{-\delta dt}.$$

Replacing e^{-rdt} with $1 - rdt$, $e^{-\delta dt}$ with $1 - \delta dt$, and eliminating terms of order $(dt)^2$ leaves the Black Scholes p.d.e.:

$$\frac{\partial}{\partial t} C(S_{T-dt}, T-dt) + \frac{\sigma^2 S_{T-dt}^2}{2} \frac{\partial^2}{\partial S^2} C(S_{T-dt}, T-dt) + (r - \delta) S_{T-dt} \frac{\partial}{\partial S} C(S_{T-dt}, T-dt) - rC(S_{T-dt}, T-dt) = 0. \quad (34)$$

Solving this p.d.e. subject to the terminal condition (33) and the appropriate boundary conditions allows us to determine $C(S_{T-dt}, T-dt)$. This allows us to set $\tau = T - dt$ and step back to the prior time $t = T - 2dt$. Obviously, this

⁶Linearity also follows from the fact that (31) implies that $dC(S_{T-dt}, T-dt)$ and dS_{T-dt} are perfectly correlated.

process can be repeated to obtain $C(S_0, 0)$. At any time t , the analog to (34) is clearly:

$$\frac{\partial}{\partial t}C(S_t, t) + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2}{\partial S^2}C(S_t, t) + (r - \delta)S_t \frac{\partial}{\partial S}C(S_t, t) - rC(S_t, t) = 0, t \in [0, T]. \quad (35)$$

From the analog to (32), the number of shares held from t to $t + dt$ is:

$$b = \frac{\partial}{\partial S}C(S_t, t), \quad t \in [0, T]. \quad (36)$$

Since this clearly varies with t , the strategy is dynamic.

The value of an American binary call at time $t < \tau_h$ is the solution $ABC(S, t)$ to the Black Scholes p.d.e. (35), restricted to the domain $S \in (0, H), t \in [0, T]$, and subject to the terminal condition:

$$ABC(S, T) = 0, \quad S \in (0, H), \quad (37)$$

and boundary conditions:

$$ABC(0, t) = 0 \quad ABC(H, t) = 1, \quad t \in [0, T]. \quad (38)$$

The solution is given in [13]. From (36), the number of shares held at $t < \tau_h$ is $\frac{\partial}{\partial S}ABC(S_t, t)$. Since this varies with t , the hedging strategy is dynamic.

To uncover a static hedge, we would like to use the linearity principle globally, as was done in the previous section. However, we have already seen in (29) that if the payoff at the first passage time is expressed as a linear combination of the stock and bond, then the value at any prior time is expressed in terms of instruments whose value is unknown by definition. A solution to this problem is to express the payoff as linear in an alternative pair of instruments whose value at any prior time is known. The pair of instruments we will use are derivative securities with a single payoff whose values have no time decay. Assuming for the moment that such securities exist, we call these instruments *stationary*, and denote their values at t by $V^s(S_t, t)$. The relationship between these values and a stationary claim's payoff is given by the following **Stationarity Principle**:

If:

$$V^s(S_\tau, \tau) = f(S_\tau), \quad (39)$$

for some stopping time τ , then

$$V^s(S_t, t) = f(S_t), \quad (40)$$

at any prior time $t \leq \tau$ for which the function $f(\cdot)$ is known.

Thus, if a stationary instrument pays off $f(S_\tau)$ at τ , then its value at any prior time t is simply $f(S_t)$. The proof is straightforward. Fix S_t at some level S and define $v(t) \equiv V^s(S, t)$ for all $t \in [0, \tau]$. Then from (39):

$$f(S) = V^s(S, \tau) \equiv v(\tau) = v(t) + \int_t^\tau v'(u)du,$$

by the Fundamental Theorem of Calculus. However, $v'(u) = \frac{\partial}{\partial u}V^s(S, u) = 0$, by stationarity. Consequently,

$$f(S) = v(t) \equiv V^s(S, t)$$

for any level of S_t , proving (40).

It is interesting to compare the Linearity Principle expressed in (13) and (14) with the Stationarity Principle given in (39) and (40). In both cases, the payoff function holding at maturity also relates the value of the derivative to the value of its underlying at any earlier time. In the former principle, which assumed zero interest rates and dividends, this payoff function is linear. In the latter principle, differentiating (30) with respect to time implies that the payoff function cannot be linear, unless $r = \delta = 0$.

To determine the exact form of these non-linear payoffs, we set the time derivative in the Black Scholes p.d.e. (35) to zero:

$$\frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} V^s(S, t) + (r - \delta) S \frac{\partial}{\partial S} V^s(S, t) - r \frac{\partial}{\partial S} V^s(S, t) = 0. \quad (41)$$

Substituting a guess that $V^s(S, t) = S^p$ yields the following quadratic equation for the power p :

$$\frac{\sigma^2}{2} p(p - 1) + (r - \delta)p - r = 0. \quad (42)$$

Using the quadratic root formula, the two solutions are $p = \gamma + \epsilon$ and $p = \gamma - \epsilon$ where:

$$\gamma \equiv \frac{1}{2} - \frac{r - \delta}{\sigma^2} \text{ and } \epsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}. \quad (43)$$

Thus, the two stationary solutions of (41) are $S^{\gamma+\epsilon}$ and $S^{\gamma-\epsilon}$, which we term the value of a pseudo-share and the value of a pseudo-bond respectively. Note that if $r = \delta = 0$, then $\gamma = \epsilon = \frac{1}{2}$ and so the pseudo-share is a share while the pseudo-bond is a bond. For arbitrary r and δ , these pseudo-securities are both non-negative and convex functions of the stock price, as shown in figures 4 and 5. Since $\gamma + \epsilon \geq 1$, the pseudo-share rises with the stock price, while since $\gamma + \epsilon \leq 0$, the pseudo-bond falls with the stock price.

Under the assumption that vanilla options of all strikes and of an arbitrarily distant⁷ maturity T_h are available, a portfolio of these options can be used to create the pseudo-share, which has a payoff of $S_{T_h}^{\gamma+\epsilon}$ at T_h . From (5) with $\kappa = 0$, the portfolio can be created out of calls alone:

$$S_{T_h}^{\gamma+\epsilon} = \int_0^\infty (\gamma + \epsilon)(\gamma + \epsilon - 1) K^{\gamma+\epsilon-2} (S_{T_h} - K)^+ dK.$$

⁷If only short-dated options are available, they can be used to form the pseudo-share, and rolled over just prior to expiration. The stationarity property implies that the rollover is self-financing. Of course the hedge is no longer purely static.

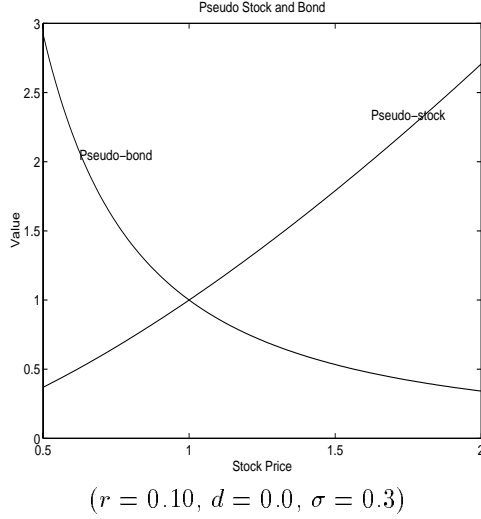


Figure 4: Figure 4: The value of a pseudo-stock and a pseudo-bond is graphed against the stock prices from 0 to 2.

Since the value of this payoff is stationary, the stationarity principle implies that the value of this portfolio at any prior time $t \leq T_h$ is simply $S_t^{\gamma+\epsilon}$. Similarly, from (5) with $\kappa \uparrow \infty$, the following portfolio of vanilla puts can be used to create the pseudo-bond:

$$S_{T_h}^{\gamma-\epsilon} = \int_0^\infty (\gamma - \epsilon)(\gamma - \epsilon - 1)K^{\gamma-\epsilon-2}(K - S_{T_h})^+ dK.$$

Once again, the prior value of the pseudo-bond is simply $S_t^{\gamma-\epsilon}$. By the linearity principle, any claim whose payoff is a linear combination of the prices of these securities at a stopping time will be priced so as to preserve this linear relationship at any prior time. This yields another two asset version of the linearity principle:

If:

$$V_\tau = aS_\tau^{\gamma-\epsilon} + bS_\tau^{\gamma+\epsilon}, \quad (44)$$

for any stopping time τ , then in the Black-Scholes model, the absence of arbitrage requires:

$$V_t = aS_t^{\gamma-\epsilon} + bS_t^{\gamma+\epsilon},$$

at any prior time $t \leq \tau$ for which the constants a and b are known.

Note that since a and b are constant, the security's value V is also stationary.

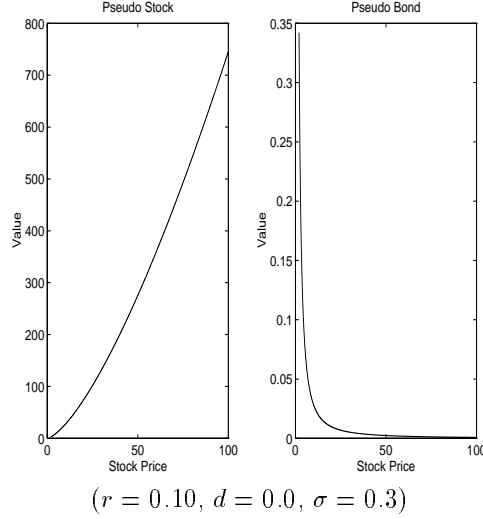


Figure 5: Figures 5a and 5b: The values of a pseudo-stock and a pseudo-bond are graphed against stock prices from 2 to 100.

4.1 Perpetual American Binary Calls

We will again consider the more general case of a perpetual double barrier option, paying R_h dollars at the hitting time if the first barrier hit is $H > S_0$, and paying R_ℓ dollars at the hitting time if the first barrier hit is $L \in (0, S_0)$. Accordingly, we set τ to be the earlier of the hitting times τ_h and τ_ℓ . Along paths for which $\tau = \tau_h$, we have:

$$R_h = V_{\tau_h} = aH^{\gamma-\epsilon} + bH^{\gamma+\epsilon}.$$

Along paths for which $\tau = \tau_\ell$, we have:

$$R_\ell = V_{\tau_\ell} = aL^{\gamma-\epsilon} + bL^{\gamma+\epsilon}.$$

Solving the two linear equations for a and b gives:

$$\begin{aligned} b^* &= \frac{R_h - R_\ell(H/L)^{\gamma-\epsilon}}{H^{\gamma+\epsilon} - H^{\gamma-\epsilon}L^{2\epsilon}} \\ a^* &= \frac{R_\ell}{L^{\gamma-\epsilon}} - b^*L^{2\epsilon}. \end{aligned} \tag{45}$$

Thus, at τ , the payoff of the perpetual double barrier option is linear in the pseudo-bond and the pseudo-stock:

$$PDBO_\tau = a^*S_\tau^{\gamma-\epsilon} + b^*S_\tau^{\gamma+\epsilon},$$

where a^* and b^* are known at any prior time $t \leq \tau$. Hence from the linearity principle (44), the time t price of a perpetual double barrier option is given by:

$$PDBO_t = a^* S_t^{\gamma-\epsilon} + b^* S_t^{\gamma+\epsilon}, \quad S_t \in (L, H), t \leq \tau. \quad (46)$$

Again setting $R_h = 1, R_\ell = 0$ and letting $L \downarrow 0$ in (45) implies $b^* = \frac{1}{H^{\gamma+\epsilon}}$ and $a^* = 0$, so that the time t value of a perpetual American binary call is given by:

$$PABC_t = \left(\frac{S_t}{H}\right)^{\gamma+\epsilon}, \quad S_t \in (0, H), t \leq \tau_h. \quad (47)$$

Consequently, at any time prior to hitting the barrier H , the payoffs of a perpetual American binary call can be replicated by buying the quantity $\frac{1}{H^{\gamma+\epsilon}}$ pseudo-shares. On paths which eventually hit the barrier, the pseudo-shares can be sold for a dollar at the hitting time. On any paths which never⁸ hit the barrier, the binary call is worthless, while the future value of the replicating portfolio is bounded above by a dollar, whose present value approaches zero for positive interest rates. This observation implies that practically⁹ any perpetual up-and-out European option, which pays a rebate of R at the knockout barrier, is valued as:

$$PUOO_t = R \left(\frac{S_t}{H}\right)^{\gamma+\epsilon}. \quad (48)$$

As shown in McKean[12] and Samuelson [15], a perpetual but otherwise vanilla American call is also given by (48), with the exercise proceeds $R = H^* - K$ and with the optimal exercise boundary:

$$H^* = \frac{\gamma + \epsilon}{\gamma + \epsilon - 1} K. \quad (49)$$

We note that if there are no dividends ($\delta = 0$), then from (43), $\gamma + \epsilon = 1$. This implies from (49) that the exercise boundary is infinite, implying no early exercise of a perpetual vanilla American call. This also implies from (47) that a perpetual American binary call can be replicated by buying $\frac{1}{H}$ shares, extending the result of the previous section to positive interest rates.

4.2 Finite-Lived American Binary Calls

As before, a perpetual American binary call is more valuable than its finite-lived counterpart. Since both options again have the same payoff along paths which hit the barrier before T , the additional value in the perpetual option again arises

⁸If the drift of the stock is less than $\frac{\sigma^2}{2}$, then there is positive probability that the barrier is never hit.

⁹If we consider a perpetual option as the limit of a finite-lived option as the maturity goes to infinity, then the requirement is that the payoff for $S_T \in [0, H]$ be bounded above.

solely from paths which avoid the barrier. Along such paths, (47) implies that the value of a perpetual option at T is:

$$PABC_T = \left(\frac{S_T}{H}\right)^{\gamma+\epsilon}, \quad S_T < H, \tau_h \geq T. \quad (50)$$

Now, this value matches the payoff from $\frac{1}{H^{\gamma+\epsilon}}$ units of an up-and-out pseudo-share-or-nothing put struck at H , whose value at t is denoted $U\&OSVNP_t$. Thus, the value at T of the finite-lived option can be obtained from that of the perpetual option by subtracting off the payoff from the position in up-and-out pseudo-share-or-nothing puts:

$$\begin{aligned} ABC_T &= PABC_T - \frac{1}{H^{\gamma+\epsilon}}U\&OSVNP_T \\ &= \frac{1}{H^{\gamma+\epsilon}}(S_T^{\gamma+\epsilon} - U\&OSVNP_T), \quad S_T < H, \tau_h \geq T, \end{aligned} \quad (51)$$

from (50). Substituting the in-out parity relation for pseudo-share-or-nothing puts:

$$\begin{aligned} ABC_T &= \frac{1}{H^{\gamma+\epsilon}}[S_T^{\gamma+\epsilon} - PSVNP_T + U\&IPSVNP_T] \\ &= \frac{1}{H^{\gamma+\epsilon}}[PSVNC_T + U\&IPSVNP_T], \quad S_T < H, \tau_h \geq T, \end{aligned} \quad (52)$$

where $PSVNC_t$ and $U\&IPSVNP_t$ are the values at $t \in [0, T]$ of a pseudo-share-or-nothing call and an up-and-in pseudo-share-or-nothing put respectively.

It follows from the linearity principle that earlier values obey this parity relation:

$$ABC_t = \frac{1}{H^{\gamma+\epsilon}}[PSVNC_t - U\&IPSVNP_t], \quad S_t < H, \tau_h \geq t. \quad (53)$$

The pseudo-share-or-nothing call has a payoff which is path-independent and consequently, this payoff can be replicated using (5). To replicate the payoff of the up-and-in pseudo-share-or-nothing put, we mimic the approach of the last section by forming a portfolio of vanilla options whose payoff lies strictly above the barrier. Consequently, along paths which avoid the barrier, the terminal value of both the up-and-in option and this portfolio are zero. Along a path that hits the barrier, we need the payoff on the portfolio to give rise to a value which matches that of the up-and-in option at the first hitting time of the barrier. At this hitting time, the up-and-in option knocks into a pseudo-share-or-nothing put. To find the payoff above the barrier whose value is that of a pseudo-share-or-nothing put, we use the following **Symmetry Principle**, developed by Carr and Chou[5]:

Consider a payoff with support below the barrier:

$$f(S_T) = \begin{cases} \phi(S_T) & \text{if } S_T \in (0, H), \\ 0 & \text{otherwise.} \end{cases}$$

Then in the Black-Scholes model, there exists a drift-adjusted reflected payoff $f^r(S_t)$ with support above H whose value matches that of f at any time t when the spot price is at H . This payoff is given by:

$$f^r(S_T) = \begin{cases} \left(\frac{S_T}{H}\right)^{2\gamma} \phi\left(\frac{H^2}{S_T}\right) & \text{if } S_T \in (H, \infty), \\ 0 & \text{otherwise.} \end{cases} \quad (54)$$

A slightly more general version of this symmetry principle is proved in Appendix 3. Letting $\phi(S_T) = S_T^{\gamma+\epsilon}$ implies that the drift-adjusted reflected payoff of the pseudo-share-or-nothing put is:

$$\begin{aligned} f^r(S_T) &= \begin{cases} \left(\frac{S_T}{H}\right)^{2\gamma} \left(\frac{H^2}{S_T}\right)^{\gamma+\epsilon} & \text{if } S_T \in (H, \infty), \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} H^{2\epsilon} S_T^{\gamma-\epsilon} & \text{if } S_T \in (H, \infty), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This payoff is that of $H^{2\epsilon}$ units of a pseudo-bond-or-nothing call. Since the payoffs of the up-and-in pseudo-stock-or-nothing put have been matched along all possible paths, absence of arbitrage requires:

$$U\&IPSVNP_t = H^{2\epsilon} PBVNC_t, \quad S_t < H, \tau_h \geq t. \quad (55)$$

Substituting this equation in (53) gives the final result:

$$ABC_t = \frac{1}{H^{\gamma+\epsilon}} PSVNC_t + \frac{1}{H^{\gamma-\epsilon}} PBVNC_t, \quad S_t < H, \tau_h \geq t. \quad (56)$$

Thus, the American binary call value is given by the cost of a static portfolio consisting of $\frac{1}{H^{\gamma+\epsilon}}$ pseudo-share-or-nothing calls and $\frac{1}{H^{\gamma-\epsilon}}$ pseudo-bond-or-nothing calls, with the latter two calls struck at H . Once again, if the stock price avoids the barrier before T , then all options expire worthless. If the stock price touches the barrier before T , then the two path-independent options can again be sold at the hitting time for a total of one dollar (see Figure 6).

Note that prior to hitting the barrier, the payoff from each of the 2 path-independent calls can be statically replicated with a portfolio of vanilla options. These calls have a payoff of $S_T^{\gamma\pm\epsilon} 1(S_T > H)$, with the plus sign holding when the underlying is the pseudo-share and the minus sign holding when the underlying is the pseudo-bond. From (6) with κ fixed at some level below the barrier, each payoff can be replicated by a static portfolio formed at t consisting of $H^{\gamma\pm\epsilon}$ bond-or-nothing calls struck at H , $(\gamma \pm \epsilon)H^{\gamma\pm\epsilon-1}$ vanilla calls struck at H , and the infinitesimal position $(\gamma \pm \epsilon)(\gamma \pm \epsilon - 1)dK$ in all vanilla calls struck above H :

$$\begin{aligned} S_T^{\gamma\pm\epsilon} 1(S_T > H) &= H^{\gamma\pm\epsilon} 1(S_T > H) + (\gamma \pm \epsilon)H^{\gamma\pm\epsilon-1}(S_T - H)^+ \\ &\quad + \int_H^\infty (\gamma \pm \epsilon)(\gamma \pm \epsilon - 1)K^{\gamma\pm\epsilon-2}(S_T - K)^+ dK. \end{aligned}$$

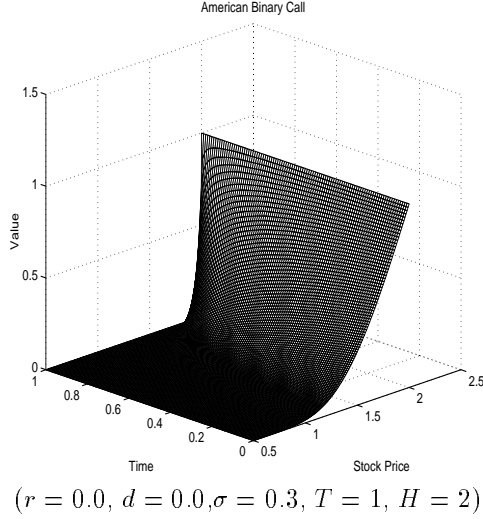


Figure 6: Figure 6: The value of an American binary call is graphed against the stock price and time.

Since the bond-or-nothing call is a vertical spread of vanilla calls, it follows from (56) that the value at t of an American binary call maturing at T can be expressed in terms of the contemporaneous prices of vanilla calls maturing at T :

$$ABC_t(T) = 2BVNC_t(T) + \frac{2\gamma}{H}C_t(H, T) + \int_H^\infty n_c(K)C_t(K, T)dK, \quad (57)$$

where:

$$n_c(K) \equiv \left[(\gamma + \epsilon)(\gamma + \epsilon - 1) \left(\frac{K}{H} \right)^{\gamma + \epsilon - 2} + (\gamma - \epsilon)(\gamma - \epsilon - 1) \left(\frac{K}{H} \right)^{\gamma - \epsilon - 2} \right] \frac{1}{H^2}.$$

We note that if $r = \delta$, then from (42), n_c simplifies to:

$$n_c(K) = \frac{2r}{\sigma^2 H^2} \left[\left(\frac{K}{H} \right)^{\gamma + \epsilon - 2} + \left(\frac{K}{H} \right)^{\gamma - \epsilon - 2} \right].$$

Thus, if $r = \delta = 0$, then $n_c = 0$ and the American binary call value is given by:

$$ABC_t(T) = 2BVNC_t(T) + \frac{1}{H}C_t(H, T).$$

Substituting $C_t(H, T) = SVN C_t(T) - HBVN C_t(T)$ gives (27).

Since the Black Scholes formula can be used to value the vanilla calls at t in terms of S_t , the American binary call value at t is also a function of S_t . It can be shown that this function $ABC(S, t)$ is in fact the solution of the Black Scholes p.d.e. (35), restricted to the domain below the barrier, and subject to (37) and (38). If this domain restriction is removed and the time t is set to maturity, then the payoff implicit in (56) is recovered:

$$ABC(S, T) = \left[\left(\frac{S}{H} \right)^{\gamma+\epsilon} + \left(\frac{S}{H} \right)^{\gamma-\epsilon} \right] 1(S > H). \quad (58)$$

As shown in Carr and Chou[5], this suggests a quick way to uncover a static hedge for an American binary *put*, paying a dollar at the first passage time to a lower barrier L . From [13], the valuation formula in terms of the stock is:

$$ABP(S, t) = \left(\frac{S}{L} \right)^{\gamma+\epsilon} N \left(\frac{\ln \left(\frac{L}{S} \right) - \epsilon \sigma^2 (T-t)}{\sigma \sqrt{T-t}} \right) + \left(\frac{S}{L} \right)^{\gamma-\epsilon} N \left(\frac{\ln \left(\frac{L}{S} \right) + \epsilon \sigma^2 (T-t)}{\sigma \sqrt{T-t}} \right), \quad (59)$$

for $S > L$ and $t < \tau_\ell$. Removing the requirements that $S > L$ and $t < \tau_\ell$ and letting $t \uparrow T$ gives a payoff of:

$$\lim_{t \uparrow T} ABP(S, t) = \left[\left(\frac{S}{L} \right)^{\gamma+\epsilon} + \left(\frac{S}{L} \right)^{\gamma-\epsilon} \right] 1(S < H).$$

From (5) with κ fixed at some level above the barrier, this path-independent payoff can be created out of vanilla puts maturing at T and struck at and below H .

Note that the value of a *perpetual* American binary put is easily obtained by setting T to infinity in (59):

$$\lim_{T \uparrow \infty} ABP(S, t) = \left(\frac{S}{L} \right)^{\gamma-\epsilon}, \quad S > L. \quad (60)$$

The stationarity of this payoff implies that the perpetual American binary put can be statically hedged using a portfolio of very long dated puts paying $\left(\frac{S_{T_h}}{L} \right)^{\gamma-\epsilon}$ at their maturity date T_h . On paths which eventually hit the barrier, these puts can be sold for a dollar at the hitting time. On any paths which never¹⁰ hit the barrier, the binary put is worthless, while the future value of the replicating portfolio is bounded above by a dollar, whose present value approaches zero for positive interest rates. If we consider a perpetual option as the limit of a finite-lived option as the maturity goes to infinity, then we can use this approach to value a perpetual down-and-out European claim. so long as

¹⁰If the drift of the stock is greater than $\frac{\sigma^2}{2}$, then there is positive probability that the barrier is never hit.

the the payoff for $S_T \in [H, \infty)$ is bounded above. If such a claim pays a rebate of R at the knockout barrier, then its value is:

$$PDDO_t = R \left(\frac{S_t}{H} \right)^{\gamma - \epsilon}. \quad (61)$$

As shown in McKean[12] and Samuelson [15], a perpetual but otherwise vanilla American put is also given by (61), with the exercise proceeds $R = K - H^*$ and with the optimal exercise boundary:

$$H^* = \frac{\gamma - \epsilon}{\gamma - \epsilon - 1} K.$$

We note that if interest rates vanish ($r = 0$), then this exercise boundary is zero, implying no early exercise of a perpetual vanilla American put. This also implies from (59) that a perpetual American binary put can be replicated by buying a bond. Note however, that when dividends vanish instead ($\delta = 0$), the perpetual American binary put cannot be statically hedged by buying $\frac{1}{L}$ shares, in contrast to the corresponding result for calls.

The next section shows that finite-lived American binary options are fundamental in that any barrier option can be valued as a static portfolio of these options.

5 Static Hedging of Barrier Options

Breeden and Litzenberger[4] showed that static positions in vanilla options can be used to uncover the state pricing density, where states are defined by terminal stock price levels. In fact, using a proof given in Carr and Madan[8], Appendix 2 shows that this state pricing density $g(Z)$ is given by the cost of a position in $\frac{1}{(\Delta K)^2}$ butterfly spreads centered at Z , as $\Delta K \downarrow 0$:

$$g_t(Z) = \left\{ \begin{array}{ll} \frac{\partial^2 P_t(Z)}{\partial K^2} \equiv \lim_{\Delta K \downarrow 0} \frac{P_t(Z - \Delta K) - 2P_t(Z) + P_t(Z + \Delta K)}{(\Delta K)^2} & \text{for } Z \leq \kappa; \\ \frac{\partial^2 C_t(Z)}{\partial K^2} \equiv \lim_{\Delta K \downarrow 0} \frac{C_t(Z - \Delta K) - 2C_t(Z) + C_t(Z + \Delta K)}{(\Delta K)^2} & \text{for } Z > \kappa. \end{array} \right\}, \quad (62)$$

where recall κ is an arbitrary constant.

By analogy, this section represents a state pricing density in terms of option prices, where states are now given by first passage times to a constant barrier. In contrast to (62), the representation given is only valid in the Black Scholes model. In common with (62), the representation allows us to uncover the static hedge associated with an arbitrary payoff defined over the relevant states. In our case, this payoff occurs at the first passage time to the barrier.

Recall that an American binary call maturing at T pays one dollar at the first passage time so long as this hitting time occurs before T . Consequently,

a calendar spread of an American binary call maturing at $T + \Delta T$ over one maturing at T delivers a dollar if and only if the first hitting time is between T and $T + \Delta T$. If an investor purchases $\frac{1}{\Delta T}$ such spreads, then as ΔT approaches zero, the payoff from the position approaches that of a delta function centered at T :

$$\lim_{\Delta T \downarrow 0} \frac{ABC_{T+\Delta T} - ABC_T}{\Delta T} = \delta(\tau_h - T).$$

Recall from (56) that in the absence of arbitrage, the cost at t of purchasing an American binary call maturing at T is:

$$ABC_t(T) = \frac{1}{H^{\gamma+\epsilon}} PSVNC_t(T) + \frac{1}{H^{\gamma-\epsilon}} PBVNC_t(T), \quad S_t < H, \tau \geq t.$$

It follows that the cost of obtaining the above delta function payoff is:

$$ABC'_t(T) = \frac{1}{H^{\gamma+\epsilon}} PSVNC'_t(T) + \frac{1}{H^{\gamma-\epsilon}} PBVNC'_t(T), \quad S_t < H, \tau \geq t,$$

where $PSVNC'_t(T)$ is the time t cost as $\Delta T \downarrow 0$ of a vanilla call portfolio with payoffs:

$$\lim_{\Delta T \downarrow 0} \frac{\left(\frac{S_{T+\Delta T}}{H}\right)^{\gamma+\epsilon} 1(S_{T+\Delta T} > H)}{\Delta T} \text{ at } T+\Delta T, \text{ and } \lim_{\Delta T \downarrow 0} -\frac{\left(\frac{S_T}{H}\right)^{\gamma+\epsilon} 1(S_T > H)}{\Delta T} \text{ at } T,$$

and similarly, $PBVNC'_t(T)$ is the time t cost as $\Delta T \downarrow 0$ of a vanilla call portfolio with payoffs:

$$\lim_{\Delta T \downarrow 0} \frac{\left(\frac{S_{T+\Delta T}}{H}\right)^{\gamma-\epsilon} 1(S_{T+\Delta T} > H)}{\Delta T} \text{ at } T+\Delta T \text{ and } \lim_{\Delta T \downarrow 0} -\frac{\left(\frac{S_T}{H}\right)^{\gamma-\epsilon} 1(S_T > H)}{\Delta T} \text{ at } T.$$

In analogy with (62), we define $\phi_t(T) \equiv ABC'_t(T)$ as the state pricing density, where states are henceforth defined by first passage times to the barrier H . This density can be observed from the market prices of calls struck above H and maturing before T . The appropriate static position in these calls allows an investor to receive an infinite payoff if the first passage time is T , and no payoff otherwise.

We now show how to statically replicate the payoffs of any (single) barrier option using a portfolio of vanilla options. For concreteness, we focus on an up-and-out put struck below the initial stock price. By in-out parity, the out-put can be statically replicated if the in-put can be:

$$UOP_t = P_t - UIP_t, \quad t \leq \tau_h.$$

To value the up-and-in-put, let $P(S, t)$ be the well-known Black Scholes European put formula:

$$P(S, t) = K e^{-r(T-t)} N(-d_2(S, t)) - S e^{-\delta(T-t)} N(-d_1(S, t)), t \in [0, T], \quad (63)$$

where:

$$-d_2(S, t) \equiv \frac{\ln(K/S)}{\sigma\sqrt{T-t}} + \gamma\sigma\sqrt{T-t}, \quad -d_1(S, t) \equiv -d_2(S, t) - \sigma\sqrt{T-t},$$

and where $N(d) \equiv \int_{-\infty}^d \frac{e^{-z^2/2}}{\sqrt{2\pi}}$ is the standard normal distribution function. Then, the up-and-in-put value at t is given by:

$$UIP_t = \int_t^T \phi_t(u)p(u)du. \quad (64)$$

where:

$$p(u) \equiv P(H, u) \quad (65)$$

is the vanilla put value at the barrier. Thus, the payoffs of the up-and-in put can be statically replicated by a portfolio of vanilla calls struck above H and maturing before T . To determine the position in each call, integrate (64) by parts:

$$UIP_t = - \int_t^T ABC_t(u)p'(u)du, \quad (66)$$

where from (63) and (65), the put's time derivative is given by:

$$\begin{aligned} p'(u) &= rKe^{-r(T-u)}N(-d_2(H, u)) - \delta He^{-\delta(T-u)}N(-d_1(H, u)) \\ &\quad - \frac{\sigma Ke^{-r(T-u)}}{2\sqrt{T-u}}N'(-d_2(H, u)), \quad u \in [t, T]. \end{aligned}$$

Combining (66) with (57) gives the position in calls:

$$UIP_t = \int_t^T (-p'(u)) \left[2BVNC_t(u) + \frac{2\gamma}{H}C_t(H, u) + \int_H^\infty n_c(K)C_t(K, u)dK \right] du. \quad (67)$$

Thus, this static hedge uses calls maturing up to T and struck at or beyond the barrier. However, if $r = \delta = 0$, then $n_c(K) = 0$ and so only calls struck at or near the barrier are used. Of course, when the integral in (67) does not vanish, it must be discretized in practice.

More generally, we may consider an up-and-out claim with a payoff of $f(S_T)$ paid at maturity if the barrier is avoided and a time-dependent rebate of $g(\tau_h)$ paid at the hitting time otherwise. Letting $V(S, t)$ denote the Black-Scholes value¹¹ at t of the payoff $f(S_t)1(S_T < H)$, the value of this up-and-out claim is given by:

$$UOV_t = V(S, t) + \int_t^T \phi_t(u)[g(u) - V(H, u)]du. \quad (68)$$

¹¹With states defined by stock price levels, the state pricing density g can be used to determine this value (see (71) in Appendix 3.). Equivalently, the expansion (6) can be used with $\kappa = H$ and the Black Scholes formula (63) used to value the vanilla puts at the barrier.

Integrating by parts gives:

$$UOV_t = V(S, t) + ABC_t(T)[g(T) - V(H, T)] - \int_t^T ABC_t(u) \left[g'(u) - \frac{\partial}{\partial u} V(H, u) \right] du, \quad (69)$$

since $ABC_t(t) = 0$ for $S_t < H$. The static hedge for the first term is obtained from (5), while the static hedge for the second and third terms can be obtained from the appropriate weighting of the static hedge for an American binary call given in (57). For down-and-out claims, one would instead use the static hedge for an American binary put.

Summary

By exploiting the principles of linearity, stationarity and symmetry, we showed how to conduct static hedges of barrier options. It would be relatively straightforward to extend the analysis to multiple barrier options and to lookback options. A much more challenging problem is to extend the analysis to arbitrarily time-dependent interest rates, dividend yields, and variance rates, while retaining the analytical flavor of the foregoing analysis. In the interests of brevity, this extension is left for future research.

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Appendix 1: Proof of (5)

The Fundamental Theorem of Calculus implies that for any fixed κ :

$$\begin{aligned} f(S) &= f(\kappa) - 1(S < \kappa) \int_S^\kappa f'(u) du + 1(S > \kappa) \int_\kappa^S f'(u) du \\ &= f(\kappa) - 1(S < \kappa) \int_S^\kappa \left[f'(\kappa) - \int_u^\kappa f''(v) dv \right] du \\ &\quad + 1(S > \kappa) \int_\kappa^S \left[f'(\kappa) + \int_\kappa^u f''(v) dv \right] du. \end{aligned}$$

Noting that $f'(\kappa)$ does not depend on u and reversing the order of integration yields:

$$f(S) = f(\kappa) + f'(\kappa)(S - \kappa) + 1(S < \kappa) \int_S^\kappa \int_S^v f''(v) du dv + 1(S > \kappa) \int_\kappa^S \int_\kappa^S f''(v) du dv.$$

Performing the integral over u yields:

$$f(S) = f(\kappa) + f'(\kappa)(S - \kappa) + 1(S < \kappa) \int_S^\kappa f''(v)(v - S) dv + 1(S > \kappa) \int_\kappa^S f''(v)(S - v) dv,$$

and this may be equivalently re-written as:

$$f(S) = f(\kappa) + f'(\kappa)(S - \kappa) + \int_0^\kappa f''(v)(v - S)^+ dv + \int_\kappa^\infty f''(v)(S - v)^+ dv.$$

Appendix 2: Proof of (62)

Integrating equation (6) by parts gives:

$$V_t = f(\kappa)B_t + f'(\kappa)I_t(\kappa) + f'(K)P_t(K) \Big|_0^\kappa - \int_0^\kappa f'(K)P_t'(K)dK + f'(K)C_t(K) \Big|_\kappa^\infty - \int_\kappa^\infty f'(K)C_t'(K)dK.$$

Since $P(0) = 0$ and $C(\infty) = 0$ and $I(\kappa) = C(\kappa) - P(\kappa)$, the second, third and fifth terms cancel. Integrating by parts once again:

$$V_t = f(\kappa)B_t - f(K)P_t'(K) \Big|_0^\kappa + \int_0^\kappa f(K)P_t''(K)dK - f(K)C_t'(K) \Big|_\kappa^\infty + \int_\kappa^\infty f(K)C_t''(K)dK.$$

Noting that $C_t'(\infty) = P_t'(0) = 0$ and $P_t'(\kappa) - C_t'(\kappa) = B_t$ by differentiating put-call-parity (12), we observe that:

$$V_t = \int_0^\infty f(K)g_t(K)dK,$$

where $g_t(K)$ is the second derivative with respect to strike of an option price:

$$g_t(K) \equiv \begin{cases} \frac{\partial^2 P_t(K)}{\partial K^2} & \text{for } K \leq \kappa; \\ \frac{\partial^2 C_t(K)}{\partial K^2} & \text{for } K > \kappa. \end{cases}$$

Appendix 3: Proof of (54)

So that one can statically hedge down-options, we prove the following slightly more general version of the Symmetry Principle:

Consider a payoff with support in the interval (A, B) :

$$f(S_T) = \begin{cases} \phi(S_T) & \text{if } S_T \in (A, B), \\ 0 & \text{otherwise.} \end{cases}$$

Then in the Black Scholes model, there exists a drift-adjusted reflected payoff $f^r(S_t)$ whose value matches that of f at any time t when the spot price is at H . This payoff is given by:

$$f^r(S_T) = \begin{cases} \left(\frac{S_T}{H}\right)^{2\gamma} \phi\left(\frac{H^2}{S_T}\right) & \text{if } S_T \in \left(\frac{H^2}{B}, \frac{H^2}{A}\right), \\ 0 & \text{otherwise.} \end{cases} \quad (70)$$

Proof. From (63) and (62) as $\kappa \uparrow \infty$, the state pricing density in the Black Scholes model is:

$$g_t(S_t, S_T) = \frac{e^{-r\tau}}{S_T \sigma \sqrt{\tau}} N' \left(\frac{\ln(S_T/S_t)}{\sigma \sqrt{\tau}} + \gamma \sigma \sqrt{\tau} \right), \quad (71)$$

where $\tau \equiv T - t$. Thus, the value of f when the spot price is at H at time t is:

$$\begin{aligned} V_f(H, t) &= \int_A^B \phi(S_T) g_t(H, S_T) dS_T \\ &= \int_A^B \phi(S_T) \frac{e^{-r\tau}}{S_T \sigma \sqrt{\tau}} N' \left(\frac{\ln(S_T/H)}{\sigma \sqrt{\tau}} + \gamma \sigma \sqrt{\tau} \right) dS_T. \end{aligned}$$

Let $\hat{S}_T = \frac{H^2}{S_T}$. Then, $dS_T = -\frac{H^2}{\hat{S}_T^2} d\hat{S}_T$ and:

$$\begin{aligned} V_f(H, t) &= - \int_{H^2/A}^{H^2/B} \phi\left(\frac{H^2}{\hat{S}_T}\right) \frac{e^{-r\tau}}{\hat{S}_T \sigma \sqrt{\tau}} N' \left(\frac{\ln(H/\hat{S}_T)}{\sigma \sqrt{\tau}} + \gamma \sigma \sqrt{\tau} \right) d\hat{S}_T \\ &= \int_{H^2/B}^{H^2/A} \left(\frac{\hat{S}_T}{H}\right)^{2\gamma} \phi\left(\frac{H^2}{\hat{S}_T}\right) \frac{e^{-r\tau}}{\hat{S}_T \sigma \sqrt{\tau}} N' \left(\frac{\ln(\hat{S}_T/H)}{\sigma \sqrt{\tau}} + \gamma \sigma \sqrt{\tau} \right) d\hat{S}_T \\ &= \int_{H^2/B}^{H^2/A} \left[\left(\frac{\hat{S}_T}{H}\right)^{2\gamma} \phi\left(\frac{H^2}{\hat{S}_T}\right) \right] g_t(H, \hat{S}_T) d\hat{S}_T \\ &= V_{f^r}(H, t). \end{aligned}$$

■

Equation (54) arises from setting $A = 0$ and $B = H$ in (70).