Deriving derivatives of derivative securities

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Various techniques are used to simplify the derivations of ''greeks'' of path-independent claims in the Black–Merton–Scholes model. First, delta, gamma, speed, and other higher-order spatial derivatives of these claims are interpreted as the values of certain quantoed contingent claims. It is then shown that all partial derivatives of such claims can be represented in terms of these spatial derivatives. These observations permit the rapid deployment of high-order Taylor series expansions, and this is illustrated for the case of European options.

1. INTRODUCTION

In spite of increasing evidence against it, the Black–Merton–Scholes (BMS) model remains the *lingua franca* of option pricing. Widely used terms such as *implied volatility* and *volatility smile* are defined only in terms of this model. Standard definitions of the so-called greeks (e.g., delta, gamma, vega[1]) also rely on the model. When other models are used in practice, the outputs of such models are routinely translated into standard BMS outputs such as implied volatility.

Given this state of affairs, a deep understanding of the BMS model is a prerequisite for meaningful interactions with practitioners. Since the mechanism by which arbitrage-free values are obtained in this model is well understood, this paper examines greeks, which enjoy multiple applications. It is well known that greeks are frequently used for hedging, market risk measurement, and profit and loss attribution. They are also used in model risk assessment and optimal contract design, and to imply out parameters. While symbolic algebra programs can derive arbitrary greeks, they cannot replace an intuitive understanding of the role, genesis, and relationships between all the various greeks.

This paper develops these relationships for path-independent claims such as European calls and puts. We recognize that path-independent claims are the simplest to value and that many listed and over-the-counter derivatives are not captured by this focus. The motivation for studying these types of contract is thus primarily as a stepping stone to more complicated contracts. In spite of the apparent simplicity in the specification of the contract payoff, the expressions we will develop for (arbitrary) partial derivatives are quite complicated. We therefore leave for future research the corresponding results for claims with more complicated payoffs.

For path-independent claims, we show that in the BMS model delta, gamma, speed, and higher-order price derivatives can always be interpreted as the value

[1] Vega is sometimes renamed kappa since it is not a Greek letter.

of a certain quantoed contingent claim. This interpetation allows one to transfer intuitions regarding values to these greeks and to apply any valuation methodology to determine them. To quickly derive these results, we use a technique first presented by Bergman (1983). Consider the arbitrage-free value function $V(S, t)$ of a claim with a final payoff of $f(S_T)$ at T. It is well known that this function solves the BMS partial differential equation (PDE):

$$\frac{\partial V(\tau, S)}{\partial \tau} = \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(\tau, S)}{\partial S^2} + (r-q)S\frac{\partial V(\tau, S)}{\partial S} - rV(\tau, S) \tag{1}$$

$$\text{for } S > 0, \tau \in (0, T),$$

$$\text{subject to:} \quad V(0, S) = f(S). \tag{2}$$

Differentiating with respect to the stock price S gives a PDE for delta $V_s(S, t)$:

$$\frac{\partial V_s(\tau, S)}{\partial \tau} = \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_s(\tau, S)}{\partial S^2} + (r-q+\sigma^2)S\frac{\partial V_s(\tau, S)}{\partial S} - qV(\tau, S) \tag{3}$$

$$\text{for } S > 0, \tau \in (0, T),$$

$$\text{subject to:} \quad V_s(0, S) = f'(S). \tag{4}$$

Thus, the delta satisfies the same PDE as the value with a modified drift $r - q + \sigma^2$ and a modified discount rate q. This procedure can be repeated indefinitely to get higher-order stock price derivatives. We illustrate this result by deriving a completely explicit formula for the nth-order price derivative of an option.

For path-independent claims in the BMS model, this paper generalizes the well-known result that theta can be expressed in terms of the first three price derivatives, i.e., value, delta, and gamma. In particular, we relate *any* partial derivative of such claims to the spatial derivatives. These relationships permit rapid deployment of Taylor series expansions, which can sometimes be faster than recomputing values. Besides providing computational advantages, these relationships allow understanding of the behavior of one greek (e.g., gamma) to be transferred to other greeks (e.g., vega).

The theoretical contribution of this paper can thus be summarized as follows. To our knowledge, we provide the first explicit expression for an "arbitrary greek" of a path-independent claim, a term which is formally defined in the body of the paper. We also first show that any such greek can be reexpressed solely in terms of stock price derivatives. We give new financial interpretations of these stock price derivatives and we introduce the use of operator calculus to quickly derive our results. Finally, we explore the use of multivariate Taylor series expansions and the restrictions under which such series converge.

The structure of this paper is as follows. Section 2 reviews previous literature on determining greeks and reviews the BMS model of contingent claim valuation. Section 3 shows how delta, gamma, speed, and higher-order price derivatives can be interpreted as the values of certain quantoed contingent claims. The following section uses operator calculus to relate first partials with respect to the dividend yield, the riskless rate, and the volatility rate to the claim's value, delta, and gamma. Section 5 generalizes these results by expressing an arbitrary greek in terms of spatial derivatives. Section 6 conducts Taylor series expansions of a call in all of its independent variables and explores the limitations of this commonly used approach. The paper concludes with a summary and a description of possible extensions. The Appendix contains some technical results.

2. LITERATURE REVIEW AND THE BMS MODEL

2.1 *Literature Review*

This section reviews the literature on determining the partial derivatives of contingent claims values. Many textbooks (e.g., Hull 1997, Chap. 14) contain short descriptions of the primary greeks, i.e., delta, gamma, vega, theta, rho, and phi (the dividend yield derivative). Pelsser and Vorst (1994) discuss the determination of these greeks in the context of the binomial model (see Cox and Rubinstein 1983). Garman (1992) christens three more partial derivatives with the names speed ($\partial^3/\partial S^3$), charm ($\partial^2/\partial S \partial t$), and color ($\partial^3/\partial S^2 \partial t$). The duration of option portfolios is defined in Garman (1985), while *volatility immunization* and *gamma duration* are defined in Garman (1999). Similarly, Haug (1993) discusses the aggregation of vegas of options of different maturities. Hull and White (1987) compare delta hedging, delta+gamma hedging, and delta+vega hedging of written FX options and conclude that the last of these works best. Willard (1997) calculates sensitivities for path-independent derivative securities in multifactor models, while Ross (1998) calculates sensitivities for multiasset European options.

Estrella (1995) derives an algorithm[2] for determining arbitrary price derivatives of the BMS option formula. He then examines Taylor series expansions in the stock price and finds the radius of convergence. Broadie and Glasserman (1996), Curran (1993), and Glasserman and Zhao (1999) all consider the estimation of security price derivatives using simulation. Bergman (1983) and Bergman, Grundy, and Wiener (1996) derive expressions for delta and gamma when volatility is a function of stock price and time. Grundy and Wiener (1996) also derive theoretical and empirical bounds on deltas for this case. Andreasen (1996) uses Dupire's (1994) forward PDE in this setting to efficiently develop price derivatives of European options numerically. Fournié *et al*. (1997) and Bermin (1999) use Malliavin calculus to determine deltas in even more general settings. There is a substantial literature on durations of bonds, which this literature survey ignores in the interests of brevity. However, Bergman (1998) and Hull and White (1992) examine greeks of interest rate derivatives in diffusive single-factor models. Similarly, using an option pricing

[2] This paper finds the solution of his recursion.

context, Ferri, Oberhelman, and Goldstein (1982) examine yield sensitivities of short-term securities, while Ogden (1987) examines yield sensitivities of corporate bonds.

In a very general context, Breeden and Litzenberger (1978) show that the second derivative with respect to an option's strike price can be used to imply out state-contingent prices. Similarly, Schroder (1995) shows that the first derivative with respect to strike of an American option yields the risk-neutral probability of exercise. He also interprets deltas of American options. In this paper, we do not consider derivatives with respect to strike price, since our focus is on general statements for European-style contingent claims, rather than on options per se.

2.2 *The BMS Model*

The BMS model assumes frictionless security markets and that, over the contingent claim's life $[0, T]$, there is a constant riskless rate r and a constant continuous dividend yield q from the underlying security, whose price S obeys geometric Brownian motion:

$$\frac{\mathrm{d}S_t}{S_t} = \alpha_t \,\mathrm{d}t + \sigma \,\mathrm{d}B_t, \quad t \in [0, T]. \tag{5}$$

As usual, the process α_t is the expected growth rate in the underlying security price, $\sigma > 0$ is the security's constant volatility rate, and $\{B_t; t \geqslant 0\}$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

Consider a path-independent claim whose final payoff $f(S)$ is a known function of S. Let $V(\tau, S)$ be a function relating the claim's arbitrage-free value V to the claim's time to maturity $\tau \equiv T - t$ and to the underlying security price S. It is well known that $V(S, \tau)$ solves the initial value problem consisting of (1) and (2). It is also well known that there exists a unique measure $Q^{(0)}$ under which the "risk-neutral" stock price process is

$$\frac{\mathrm{d}S_t}{S_t} = (r - q)\,\mathrm{d}t + \sigma \,\mathrm{d}B_t^{(0)}, \quad t \in (0, T), \tag{6}$$

where $\{B_t^{(0)}; t \in (0, T)\}$ is a $Q^{(0)}$ standard Brownian motion. Under the measure $Q^{(0)}$, the forward price of the underlying and the forward price of any path-independent claim $\mathrm{e}^{r(T-t)} V(T - t, S_t)$ are both martingales. Consequently, the solution of (1) and (2) can be expressed as

$$V(\tau, S) = \mathrm{e}^{-r\tau} \mathsf{E}^{(0)}[f(S_T) \mid S_t = S], \qquad S > 0, \quad \tau \in (0, T). \tag{7}$$

For certain payoff functions (e.g., $f(S) = \max\{0, S - K\}$), the integral implicit in (7) can be expressed in terms of special functions (e.g., the normal distribution function). However, for a general payoff function (e.g., the payoff function is a rational function of S), the integral implicit in (7) must be done numerically. In such cases, the computation of higher-order partial derivatives can be numerically intensive and so there are computational motivations for exploring relationships between these greeks.

3. STOCK PRICE DERIVATIVES AND QUANTOING

This section develops expressions for delta, gamma, speed, and higher-order derivatives with respect to the stock price. The next two sections show that one use of these spatial derivatives is as an input to the calculation of other partial derivatives. While only the first price derivative is needed to perfectly hedge a contingent claim in the model, it can be shown that gamma governs the hedging error when hedging continuously at the wrong volatility in a diffusion setting. Furthermore, gamma, speed, and other higher-order derivatives govern the hedging error when prices jump and/or when rebalancing is discrete. All of these price greeks are needed to do a Taylor series expansion of the value V in S. This section shows that every price derivative can be regarded as the arbitrage-free value of a certain quantoed contingent claim. This interpretation allows intuition regarding values to be transferred to these greeks. It also allows any methodology for determining values to be applied to determining these greeks.

To simplify calculations, we first transform the BMS PDE (1) by expressing the value V in terms of the log of the stock price. Let

$$U(\tau, x) \equiv V(\tau, S), \tag{8}$$

where $x \equiv \ln S$. Then $\partial V(\tau, S)/\partial\tau$ on the left-hand side of (1) can be replaced by $\partial U(\tau, x)/\partial\tau$. For the right-hand side, we use the following general relationship between a stock price derivative and log price derivatives:

$$S^{l_s} D_S^{l_s} = \sum_{i_s=1}^{l_s} \mathcal{S}_1(l_s, i_s) D_x^{i_s}, \quad l_s = 1, 2, \ldots, \tag{9}$$

where $\mathcal{S}_1(l_s, i_s)$ denotes a Stirling number of the first kind.[3] Using (8) for $l_s = 1, 2$ implies that

$$S\frac{\partial V(\tau, S)}{\partial S} = \frac{\partial U(\tau, x)}{\partial x}, \tag{10}$$

$$S^2 \frac{\partial^2 V(\tau, S)}{\partial S^2} = -\frac{\partial U(\tau, x)}{\partial x} + \frac{\partial^2 U(\tau, x)}{\partial x^2}. \tag{11}$$

Substituting into the BMS PDE (1) yields a PDE with constant coefficients:

$$\frac{\partial U(\tau, x)}{\partial \tau} = \tfrac{1}{2}\sigma^2 \frac{\partial^2 U(\tau, x)}{\partial x^2} + \mu \frac{\partial U(\tau, x)}{\partial x} - rU(\tau, x), \qquad x \in \mathbb{R}, \quad \tau \in (0, T), \tag{12}$$

[3] These numbers satisfy a simple recursion and are given by a complicated closed-form solution in Appendix 4.

where $\mu \equiv r - q - \frac{1}{2}\sigma^2$. Under (8), the initial condition (2) transforms to

$$U(0, x) = f(e^x) \equiv \phi(x). \tag{13}$$

Let D_x^l denote the lth derivative with respect to x. Differentiating (12) l times with respect to x implies that $D_x^l U(\tau, x)$ satisfies the same PDE as $U(\tau, x)$. Consequently, the process $e^{r(T-t)} D_x^l U(T-t, X_t)$ is a $Q^{(0)}$ martingale for $l = 0, 1, 2, \ldots$, where $X_t \equiv \ln S_t$. It follows that $D_x^l U(T-t, X_t)$ can be interpreted as the value at t of a contingent claim with the single payoff $\phi^{(l)}(X_T)$ occurring at T.

For example, when $l = 1$, $U_x(T-t, X_t) = S_t V_s(T-t, S_t)$ is the value at t of a claim paying $\phi'(X_T) = S_T f'(S_T)$ at T. In the case of a call, $f(S) = \max\{0, S-K\}$, and so the payoff associated with U_x is that of a gap call $S_T \mathbb{1}(S_T > K)$. Clearly, $U_x(T-t, X_t) = S_t V_s(T-t, S_t)$ can also be interpreted as the dollar amount invested in the stock at time t when dynamically replicating the payoff $f(S_T)$ occurring at T. For $l = 2$, equations (10) and (11) imply that

$$U_{xx}(T-t, X_t) = S_t^2 V_{ss}(T-t, S_t) + S_t V_s(T-t, S_t), \tag{14}$$

and so $U_{xx}(T-t, X_t)$ is the value at t of a claim paying $\phi''(X_T) = S_T^2 f''(S_T) + S_T f'(S_T)$ at T. Since $S_t V_s(T-t, S_t)$ is also a claim price process, (14) implies that $S_t^2 V_{ss}(T-t, S_t)$ is yet another claim price process, with payoff $S_T^2 f''(S_T)$ at T. Furthermore, the process $U_{xx}(T-t, X_t)$ can also be interpreted as the dollar amount invested in the stock at time t when dynamically replicating the payoff[4] $S_T f'(S_T)$ occurring at T. Thus, for the call example, $U_{xx}(T-t, X_t)$ is the dollar amount invested in the stock at t when dynamically replicating the gap call payoff occurring at T.

The inverse of (8) giving the general relationship between a log price derivative and stock price derivatives is given by

$$D_x^{l_x} = \sum_{i_x}^{l_x} \mathcal{S}_2(l_x, i_x) S^{i_x} D_s^{i_x}, \tag{15}$$

where $\mathcal{S}_2(l_s, i_s)$ denotes a Stirling number of the second kind.[5] Using this expression, it is not hard to prove the following lemma:

Lemma 1 *For each $l = 0, 1, 2 \ldots$, the process $\{S_t^l D_s^l V(S_t, T-t); t \in [0, T]\}$ is the arbitrage-free value at t of a claim with payoff $S_T^l f^{(l)}(S_T)$ at T.*

We can use this lemma to determine the process followed by the delta $V_s(\tau, S)$ of a contingent claim, which is defined as the first partial derivative of the claim's value function $V(\tau, S)$ with respect to the price S. Since $S_t V_s(T-t, S_t)$ is the price of a claim paying $S_T f'(S_T)$ at T, it follows that delta can be interpreted as the price in shares of a claim paying $f'(S_T)$ shares at T. Thus, for the call

[4] By induction, $D_x^l U(T-t, X_t)$ is the dollar amount invested in the stock at t when dynamically replicating the payoff $\phi^{(l-1)}(X_T)$ at T.

[5] These numbers satisfy a simple recursion and are given by a simple closed-form solution in Appendix 4.

example, delta is the price in shares of a claim paying one share if $S_T > K$ at T. Since the function $V_s(\tau, S)$ relates the price in shares to the price of the stock *in dollars*, we are in the same situation as when the payoff of an option is defined in terms of a different currency than the price of the underlying. This option is said to be quantoed and it is well known that a so-called "quanto correction" is required in specifying the risk-neutral process. When the currency being quantoed into is a share, the quanto correction for the stock price results in the following modification of the "risk-neutral" stock price process (6):

$$\frac{\mathrm{d}S_t}{S_t} = (r - q + \sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^{(1)}, \quad t \in [0, T], \tag{16}$$

where $\{B_t^{(1)}; t \in [0, T]\}$ is a $Q^{(1)}$ standard Brownian motion. The appropriate discount rate for discounting share-denominated payoffs is the dividend yield q. Thus, delta can be represented as

$$V_s(\tau, S) = \mathrm{e}^{-q\tau}\mathsf{E}^{(1)}[f^{(1)}(S_T) \mid S_t = S], \qquad S > 0, \quad \tau \in (0, T), \tag{17}$$

where the operator $\mathsf{E}^{(1)}$ indicates that the expectations are calculated using (16).

The gamma $V_{ss}(\tau, S)$ of a contingent claim is defined as the second partial derivative of the claim's value $V(\tau, S)$ with respect to the price S. Since Lemma 1 implies that $S_t^2 V_{ss}(T - t, S_t)$ is the price of a claim paying $S_T^2 f''(S_T)$ at T, it follows that gamma can be interpreted as the price in "squares" of a claim paying $f''(S_T)$ squares at T. By a square, we mean a dividend-paying claim whose value is S_t^2 for all $t \in [0, T]$. To determine this dividend, one replaces $V(\tau, S)$ with S^2 on the right-hand side of (1). The resulting dividend is $(r - 2q + \sigma^2)S^2$ for a constant dividend yield of $r - 2q + \sigma^2$. Thus, for the call example, gamma is the price in squares of a claim paying $f''(S) = \delta(S_T - K)$ squares at T, where $\delta(\cdot)$ is the Dirac delta function.[6]

The appropriate discount rate for discounting square-denominated payoffs is the dividend yield $r - 2q + \sigma^2$. It follows that the gamma of a claim can be represented as

$$V_{ss}(\tau, S) = \mathrm{e}^{(r-2q+\sigma^2)\tau}\mathsf{E}^{(2)}[f''(S_T) \mid S_t = S], \qquad S > 0, \quad \tau \in (0, T), \tag{18}$$

where the operator $\mathsf{E}^{(2)}$ indicates that the expectation of the final gamma $f''(S_T)$ is calculated from the geometric Brownian motion

$$\frac{\mathrm{d}S_t}{S_t} = (r - q + 2\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^{(2)}, \quad t \in [0, T],$$

where $\{B_t^{(2)}; t \in [0, T]\}$ is a $Q^{(2)}$ standard Brownian motion.

Higher-order derivatives with respect to the underlying security price can be obtained analogously. Appendices 1 and 2 prove the following general theorem.

[6] The Dirac delta function is a generalized function characterized by two properties:

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1.$$

THEOREM 1 *The value, delta, gamma, and higher-order derivatives of path-independent claims in the BMS model are given by*

$$\frac{\partial^j V(\tau, S)}{\partial S^j} = \mathrm{e}^{[(j-1)r - jq + \frac{1}{2}(j-1)j\sigma^2]\tau} \mathsf{E}^{(j)}[f^{(j)}(S_T) \mid S_t = S] \quad (j = 0, 1, \ldots),$$

$$\text{for } S > 0, \tau \in (0, T),$$

where the operator $\mathsf{E}^{(j)}$ *indicates that the expectation of the final* j*-th derivative* $f^{(j)}(S_T)$ *is calculated from the geometric Brownian motion*

$$\frac{\mathrm{d}S_t}{S_t} = (r - q + j\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^{(j)}, \quad t \in [0, T], \tag{20}$$

and where $\{B_t^{(j)}; t \in [0, T]\}$ *is a* $Q^{(j)}$ *standard Brownian motion.*

The discount rate $-[(j-1)r - jq + \frac{1}{2}(j-1)j\sigma^2]$ in (19) is the dividend yield on a "power claim" whose value is S_t^j for all $t \in [0, T]$. The measure $Q^{(j)}$ describes prices of Arrow–Debreu securities in terms of these power claims. The payoffs of these Arrow–Debreu securities are indexed over paths and also pay out in power claims. Since the stock price S generating the paths is still denominated in dollars, a quanto correction is needed, which involves adding $j\sigma^2$ to the proportional drift in (20).

3.1 *European Options*

To illustrate higher-order price derivatives in the case of an option, let c be a call/put indicator:

$$c = \begin{cases} 1 & \text{if call,} \\ -1 & \text{if put.} \end{cases}$$

Recalling that $x \equiv \ln S$, the BMS formula for European option value is

$$\mathrm{eo}(x) \equiv c[\mathrm{e}^{x - q\tau} N(cd_1) - K\mathrm{e}^{-r\tau} N(cd_2)],$$

where

$$d_2 \equiv \frac{x - \ln K + \mu\tau}{\sigma\sqrt{\tau}}, \qquad d_1 = d_2 + \sigma\sqrt{\tau},$$

and N is the normal distribution function, given by

$$N(d) \equiv \int_{-\infty}^{d} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,\mathrm{d}z.$$

THEOREM 2 *For* $l_x = 1, 2, \ldots,$ *the* l_x*-th derivative with respect to* x *of a European option is*

$$D_x^{l_x}\mathrm{eo}(x) = c\mathrm{e}^{x - q\tau} N(cd_1) + K\mathrm{e}^{-r\tau} \frac{N'(d_2)}{\sigma\sqrt{\tau}} \sum_{i_x=0}^{l_x-2} \frac{H_{i_x}(d_2)}{(-\sigma\sqrt{\tau})^{i_x}}, \quad l_x = 1, 2, \ldots, \tag{21}$$

where $H_i(d)$ $(i = 0, 1, 2)$ *are the Hermite polynomials.*

The Hermite polynomials satisfy the recursion

$$H_{i+1}(d) = dH_i(d) - iH_{i-1}(d), \quad \text{with } H_0(d) = 1,\, H_1(d) = d.$$

The closed-form formula for the ith Hermite polynomial is well known to be given by

$$H_i(d) = \sum_{g=0}^{[i/2]} \frac{d^{i-2g}}{(i-2g)!} \frac{i!}{g!\,(-2)^g}.$$

To obtain higher-order price derivatives of an option, substitute (21) in (8). The details are left to the reader.

4. RATE DERIVATIVES AND OPERATOR CALCULUS

This section uses operator calculus to express derivatives with respect to the dividend yield, risk-free rate, and volatility in terms of the stock price derivatives derived in the last section. These derivatives are used to approximate the model risk arising from assuming that these parameters are constant over time. The volatility derivative (vega) is also often used to calculate implied volatility numerically.

The initial value problem (1, 2) governing $V(\tau, S)$ can be rewritten as

$$\frac{\partial V(\tau, S)}{\partial \tau} = \mathcal{L}[V(\tau, S)], \quad \tau > 0, \tag{22}$$

$$\text{subject to:} \quad V(0, S) = f(S), \tag{23}$$

where $\mathcal{L}$ is the following linear operator:

$$\mathcal{L} \equiv \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - q)S\frac{\partial}{\partial S} - r\mathcal{I}, \quad S > 0. \tag{24}$$

Operator calculus treats (22) as an *ordinary* differential equation in τ, by treating $\mathcal{L}$ as a constant and S as a fixed parameter. The formal solution of (22) subject to (23) is then given as

$$V(\tau, S) = \exp\{\tau \cdot \mathcal{L}\} f(S), \qquad S > 0, \quad \tau \in (0, T), \tag{25}$$

where

$$\exp\{\tau \cdot \mathcal{L}\} \equiv \sum_{j=0}^{\infty} \frac{(\tau\mathcal{L})^j}{j!}. \tag{26}$$

To justify this representation, consider a Taylor series expansion of $V(\tau, S)$ in τ

about $\tau = 0$:[7]

$$V(\tau, S) = V(0,S) + \tau \frac{\partial V(0,S)}{\partial \tau} + \frac{\tau^2}{2}\frac{\partial^2 V(0,S)}{\partial \tau^2} + \cdots + \frac{\tau^j}{j!}\frac{\partial^j V(0,S)}{\partial \tau^j} + \cdots$$
$$= \left[1 + \tau\mathcal{L} + \frac{\tau^2}{2}\mathcal{L}^2 + \cdots + \frac{\tau^j}{j!}\mathcal{L}^j + \cdots\right] V(0,S), \qquad S > 0, \quad \tau > 0, \quad (27)$$

where $\mathcal{L}^j$ is the j-fold composite of $\mathcal{L}$, i.e.,

$$\mathcal{L}^j[V(0,S)] \equiv \underbrace{\mathcal{L} \circ \mathcal{L} \circ \cdots \circ \mathcal{L}}_{j \text{ times}}[V(0,S)].$$

Substituting (23) and (26) in (27) yields (25).

Treating $\mathcal{L}$ as a constant and differentiating (25) with respect to τ recovers (22). If we analogously differentiate the formal solution (25) with respect to the dividend yield q, the riskless rate r, or the volatility σ, we rapidly obtain useful and intuitive representations of partial derivatives with respect to these variables:

$$\frac{\partial V(\tau,S)}{\partial q} = \tau \frac{\partial \mathcal{L}}{\partial q}\exp\{\tau \cdot \mathcal{L}\} f(S) = -\tau S \frac{\partial}{\partial S} V(\tau, S) \quad (28)$$

$$\frac{\partial V(\tau,S)}{\partial r} = \tau \frac{\partial \mathcal{L}}{\partial r}\exp\{\tau \cdot \mathcal{L}\} f(S) = -\tau \left[V(\tau,S) - S\frac{\partial}{\partial S} V(\tau, S)\right] \quad (29)$$

$$\frac{\partial V(\tau,S)}{\partial \sigma} = \tau \frac{\partial \mathcal{L}}{\partial \sigma}\exp\{\tau \cdot \mathcal{L}\} f(S) = \sigma\tau S^2 \frac{\partial^2}{\partial S^2} V(\tau, S), \qquad S > 0, \quad \tau > 0. \quad (30)$$

We prove that these manipulations are correct in Appendix 3.

The three results (28)–(30) are easily justified using risk-neutral valuation. To understand the result (28) for the dividend yield derivative (phi), we note that a small shift upward in the dividend yield lowers the risk-neutral mean of the terminal stock price by an amount which increases with the time to maturity (tenor). The effect of this small upward shift is thus similar to the effect of a small proportional shift downward in the initial stock price S. However, since the effect of a given downward shift on the risk-neutral mean is independent of tenor, the mimicking downward shift must be multiplied by time to maturity. Hence, up to sign, the dividend yield derivative $\partial V(\tau, S)/\partial q$ is the product of the tenor τ and the dollar investment $S\,\partial V(\tau, S)/\partial S$ in the underlying security in the replicating portfolio. The smaller this tenor and/or investment, the lower is the sensitivity to dividend yield. Thus, risk managers contemplating the design of a contingent claim can only lower phi by shortening the tenor or by lowering the expected[8] payoff delta, which will lower the initial delta. A delta-neutral portfolio of equal maturity claims is thus automatically immunized against shifts in the dividend yield. However, a short call hedged by delta shares is not

[7] For a call struck at K, all time derivatives become unbounded as $\tau \downarrow 0$ at $S = K$. Nonetheless, the right-hand side of (25) is well defined (see Hirschman and Widder 1955, p. 5).

[8] The results of the previous section implies that $Q^{(1)}$ is the measure used in computing expectations.

phi-neutral, since the dividend-paying stock should be considered as a portfolio of claims with many maturities.

Similarly, to understand the result (29) for the risk-free rate derivative (rho), we note that a small shift upward in the risk-free rate raises the risk-neutral mean of the terminal stock price by an amount which increases with tenor. By analogy with dividend yields, this effect can be mimicked by alternatively raising the initial stock price by a proportional amount, which increases with time to maturity. Since the risk-free rate is also used to discount the expected claim payoff, an upward shift in the risk-free rate also lowers the claim value by the duration τ. Hence, up to sign, a claim's rho $\partial V(\tau, S)/\partial r$ is the product of the tenor τ and the dollar investment $V(\tau, S) - S\,\partial V(\tau, S)/\partial S$ in the riskless asset in the replicating portfolio. The smaller this tenor and/or investment, the lower is the sensitivity to the riskless rate. Thus, risk managers contemplating the design of a contingent claim can only lower this sensitivity by shortening the tenor or by lowering the expected fixed component of the terminal payoff, which will lower the implicit initial investment in the riskless asset. A costless delta-neutral portfolio of equal maturity claims is immunized against shifts in the (assumed flat) term structure of interest rates. For example, a short call hedged by being long $e^{-q\tau}N(d_1)$ shares and short $KN(d_2)$ pure discount bonds is immunized, since the shares have no rho.

Finally, to understand the result (30) for the volatility derivative (vega), we note that a small shift upward in the volatility rate σ raises the standard deviation of the terminal stock price by an amount which increases with tenor. Focusing on convex payoffs, this effect can be mimicked by increasing the expected terminal gamma of the payoff, which raises the initial gamma. The larger the tenor, the more the gamma must be raised. The result (30) implies that, for a given spot price and volatility, vega depends only on the product of gamma and tenor. The smaller the tenor and/or gamma, the lower is the sensitivity to volatility. Risk managers contemplating the design of contingent claims can only lower vega by shortening the tenor or lowering convexity. A gamma-neutral portfolio of equal maturity claims must have zero vega. For example, a collar which is initially gamma-neutral will also be vega-neutral.

5. ARBITRARY GREEKS

Just as phi, rho, and vega are all simple functions of a claim's value, delta, and gamma, the BMS PDE (1) implies that a claim's time derivative (theta) can also be expressed in terms of the first three spatial derivatives. This section generalizes these results by expressing an *arbitrary greek* in terms of its *spatial derivatives*. By an arbitrary greek, we mean a partial derivative of the form $D_q^{l_q} D_r^{l_r} D_t^{l_t} D_\sigma^{l_\sigma} D_s^{l_s} V$, where D_q, D_r, D_t, D_σ, D_s denote the first-order derivative operators with respect to q, r, t, σ, S, respectively, and where $V(q, r, t, \sigma, S)$ denotes the BMS value of a path-independent claim when considered as a function of these five variables. By a spatial derivative, we mean a partial derivative with respect to the log stock price. If a representation in terms of stock price derivatives is desired, then (15) can be used with the results that follow. However, the representation in terms of derivatives with respect to log stock prices is convenient if finite differences are used to approximate the solution to the initial value problem (12, 13).

THEOREM 3 *For* $S = e^x$ *and* l_q, l_r, l_t, l_σ, l_s *positive integers, the following operators are equivalent when applied to solutions of the initial value problem* (12, 13):

$$D_q^{l_q} D_r^{l_r} D_t^{l_t} D_\sigma^{l_\sigma} D_s^{l_s} =$$

$$(-\tau)^{l_q+l_r-l_t} \frac{l_r!\,l_t!\,l_\sigma!}{(2\sigma)^{l_\sigma} S^{l_s}} \sum_{i_r=0}^{l_r} \frac{(-1)^{i_r}}{i_r!\,(l_r-i_r)!} \sum_{i_t=0}^{l_t} \frac{(-r\tau)^{i_t}}{(l_t-i_t)!}$$

$$\times \sum_{i_\sigma=\max\{[\frac{1}{2}l_\sigma+\frac{1}{2}],\,l_t-l_r-l_q-i_t\}}^{l_\sigma} \frac{(-2\sigma^2\tau)^{i_\sigma} i_\sigma!}{(l_\sigma-i_\sigma)!\,(2i_\sigma-l_\sigma)!} (l_r+l_q+i_\sigma)^{\underline{l_t-i_t}} \sum_{i_s=0}^{l_s} \mathcal{S}_1(l_s, i_s)$$

$$\times \sum_{h_t=0}^{i_t} \frac{(-\mu/r)^{h_t}}{h_t!} \sum_{h_\sigma=0}^{i_\sigma} \frac{(-1)^{h_\sigma}}{h_\sigma!\,(i_\sigma-h_\sigma)!} \sum_{g_t=0}^{i_t-h_t} \frac{(-\sigma^2/2r)^{g_t}}{g_t!\,(i_t-h_t-g_t)!} D_x^{l_q+i_r+i_\sigma+i_s+h_t+h_\sigma+2g_t}. \quad (31)$$

In this formula, $[x]$ denotes the largest integer less than or equal to x. If we let m and n denote nonnegative integers, then the notation $m^{\underline{n}}$ denotes the falling factorial, i.e.,

$$m^{\underline{n}} = \underbrace{m(m-1)\times\cdots\times m-n+1}_{n\text{ factors}}.$$

To relate an arbitrary greek to a stock price derivative, one substitutes (15) in (9).[9]

6. TAYLOR SERIES

By standard calculus, the Taylor series expansion of $V(q, r, t, \sigma, S)$ about the point $(q_0, r_0, t_0, \sigma_0, S_0)$ is

$$V(q, r, t, \sigma, S) = V(q_0, r_0, t_0, \sigma_0, S_0)$$
$$+ \sum_{m=1}^{\infty} \sum_{l_q=0}^{m} \sum_{l_r=0}^{m-l_q} \sum_{l_t=0}^{m-l_q-l_r} \sum_{l_\sigma=0}^{m-l_q-l_r-l_t} \frac{m!\,D_q^{l_q} D_r^{l_r} D_t^{l_t} D_\sigma^{l_\sigma} D_s^{l_s}}{l_q!\,l_r!\,l_t!\,l_\sigma!\,l_s!} \frac{(\Delta q)^{l_q}(\Delta r)^{l_r}(\Delta t)^{l_t}(\Delta\sigma)^{l_\sigma}(\Delta S)^{l_s}}{m!}$$
$$\times V(q_0, r_0, t_0, \sigma_0, S_0), \quad (32)$$

where $l_s \equiv m - l_q - l_r - l_t - l_\sigma$ and $\Delta q \equiv q - q_0$, $\Delta r \equiv r - r_0$, $\Delta t \equiv t - t_0$,

[9] The following relationship should be used to simplify the result:

$$\sum_{i_s} |\mathcal{S}_1(l_s, i_s)| \mathcal{S}_2(l_s, i_s)(-1)^{l_s-i_s} = \delta_{l_s=h_s},$$

where δ is the Kronecker delta function.

$\Delta\sigma \equiv \sigma - \sigma_0$, $\Delta S \equiv S - S_0$. Substituting (31) in (32) and simplifying the resulting expression gives the Taylor series expansion of a path-independent claim value in all five variables:

$$
\begin{aligned}
V(q,r,t,\sigma,S) &= V(q_0,r_0,t_0,\sigma_0,S_0)\\
&+\sum_{m=1}^{\infty}\sum_{l_q=0}^{m}\frac{(-\tau\Delta q)^{l_q}}{l_q!}\sum_{l_r=0}^{m-l_q}(-\tau\Delta r)^{l_r}\sum_{i_r=0}^{l_r}\frac{(-1)^{i_r}}{i_r!\,(l_r-i_r)!}\sum_{l_t=0}^{m-l_q-l_r}\left(\frac{\Delta t}{-\tau}\right)^{l_t}\\
&\times\sum_{i_t=0}^{l_t}\frac{(-r\tau)^{i_t}}{(l_t-i_t)!}\sum_{h_t=0}^{i_t}\frac{(-\mu/r)^{h_t}}{h_t!}\sum_{g_t=0}^{i_t-h_t}\frac{(-\sigma^2/2r)^{g_t}}{g_t!\,(i_t-h_t-g_t)!}\sum_{l_\sigma=0}^{m-l_q-l_r-l_t}\left(\frac{\Delta\sigma}{2\sigma}\right)^{l_\sigma}\frac{(\Delta S/S)^{l_s}}{l_s!}\\
&\times\sum_{i_\sigma=\max\{[\frac{1}{2}l_\sigma+\frac{1}{2}],\,l_t-l_r-l_q-i_t\}}^{l_\sigma}\frac{(-2\sigma^2\tau)^{i_\sigma}}{(l_\sigma-i_\sigma)!}\frac{i_\sigma!}{(2i_\sigma-l_\sigma)!}(l_r+l_q+i_\sigma)^{\underline{l_t-i_t}}\\
&\times\sum_{h_\sigma=0}^{i_\sigma}\frac{(-1)^{h_\sigma}}{h_\sigma!\,(i_\sigma-h_\sigma)!}\sum_{i_s=0}^{l_s}\mathcal{S}_1(l_s,i_s)D_x^{l_q+i_r+i_\sigma+i_s+h_t+h_\sigma+2g_t}U(q_0,r_0,t_0,\sigma_0,x_0),
\end{aligned}
\tag{33}
$$

where $U(q,r,t,\sigma,x) \equiv V(q,r,t,\sigma,S)$.

6.1 *European Options*

Substituting (21) in (33) yields the Taylor series expansion of the European option value $\mathrm{EO}(q,r,t,\sigma,S) \equiv \mathrm{eo}(\ln S)$ about the point $(q_0,r_0,t_0,\sigma_0,S_0)$. This formula simplifies upon observing that $\sum_{i_s=0}^{l_s}\mathcal{S}_1(l_s,i_s)=0$ for $l_s \geqslant 2$. The upper limit l_s in the last sum in (33) equals 1 only when $m=1$ and $l_q=l_r=l_t=l_\sigma=0$. In this case, the nested sum in (33) simplifies to $(\Delta S/S)D_x = c\mathrm{e}^{-q_0\tau_0}N(cd_{10})\Delta S$, where $\tau_0 \equiv T-t_0$ and

$$
d_{10} \equiv d_{20}+\sigma_0\sqrt{\tau_0}, \qquad d_{20} \equiv \frac{\ln(S_0/K)+\mu_0\tau_0}{\sigma_0\sqrt{\tau_0}}, \qquad \mu_0 \equiv r_0-q_0-\tfrac{1}{2}\sigma_0^2.
$$

Thus, the final result for the Taylor series expansion of the European option value is

$$
\begin{aligned}
\mathrm{EO}(q,r,t,\sigma,S) &= \mathrm{EO}(q_0,r_0,t_0,\sigma_0,S_0)+c\mathrm{e}^{-q_0\tau_0}N(cd_{10})\Delta S\\
&+K\mathrm{e}^{-r_0\tau_0}\frac{N'(d_{20})}{\sigma_0\sqrt{\tau_0}}\sum_{m=1}^{\infty}\sum_{l_q=0}^{m}\frac{(-\tau_0\Delta q)^{l_q}}{l_q!}\sum_{l_r=0}^{m-l_q}(-\tau\Delta r)^{l_r}\sum_{i_r=0}^{l_r}\frac{(-1)^{i_r}}{i_r!\,(l_r-i_r)!}\sum_{l_t=0}^{m-l_q-l_r}\left(\frac{\Delta t}{-\tau_0}\right)^{l_t}\\
&\times\sum_{i_t=0}^{l_t}\frac{(-r_0\tau_0)^{i_t}}{(l_t-i_t)!}\sum_{h_t=0}^{i_t}\frac{(-\mu_0/r_0)^{h_t}}{h_t!}\sum_{g_t=0}^{i_t-h_t}\frac{(-\sigma_0^2/2r_0)^{g_t}}{g_t!\,(i_t-h_t-g_t)!}\sum_{l_\sigma=0}^{m-l_q-l_r-l_t}\left(\frac{\Delta\sigma}{2\sigma_0}\right)^{l_\sigma}\frac{(\Delta S/S_0)^{l_s}}{l_s!}\mathbb{1}(l_s\neq 1)\\
&\times\sum_{i_\sigma=\max\{[\frac{1}{2}l_\sigma+\frac{1}{2}],\,l_t-l_r-l_q-i_t\}}^{l_\sigma}\frac{(-2\sigma_0^2\tau_0)^{i_\sigma}}{(l_\sigma-i_\sigma)!}\frac{i_\sigma!}{(2i_\sigma-l_\sigma)!}(l_r+l_q+i_\sigma)^{\underline{l_t-i_t}}\\
&\times\sum_{h_\sigma=0}^{i_\sigma}\frac{(-1)^{h_\sigma}}{h_\sigma!\,(i_\sigma-h_\sigma)!}\sum_{i_s=0}^{l_s}\mathcal{S}_1(l_s,i_s)\sum_{i_x=0}^{l_q+i_r+i_\sigma+i_s+h_t+h_\sigma+2g_t-2}\frac{H_{i_x}(d_{20})}{(-\sigma_0\sqrt{\tau_0})^{i_x}}.
\end{aligned}
\tag{34}
$$

Figure 1 shows a well-behaved expansion of a European call value at a point

$$(q_1, r_1, t_1, \sigma_1, S_1) = (.05, .1, .25, .25, 110)$$

about

$$(q_0, r_0, t_0, \sigma_0, S_0) = (.02, .06, 0, .2, 100)$$

for $(K, T) = (100, 1)$. As the order is increased from 1 to 4, the truncated Taylor series gets closer to the correct value until the specified tolerance of .01 is achieved. The example illustrates that high-order truncated expansions are sometimes needed to achieve the desired result.

Figure 2 focuses on a univariate Taylor series expansion in volatility, holding (q, r, t, S, K, T) constant at $(.02, .06, 0, 100, 100, 1)$. The left panel shows that even an eighth-order truncated Taylor series expansion is insufficient if the volatility changes by a sufficient magnitude. However, the right panel shows that, for the small change in volatility from .2 to .25, convergence occurs rapidly.

Since truncated Taylor series are sometimes used in place of a recalculation, it is worth investigating the region of convergence for the five independent variables. The following theorem holds for any payoff function.

THEOREM 4 *Let* $\rho_y \geqslant 0$ *denote the radius of convergence in the variable y of the Taylor series expansion of claim value V about the point* $(q_0, r_0, t_0, \sigma_0, S_0)$. *Then* $\rho_r = \infty$, $\rho_q = \infty$, $\rho_s = S_0$, $\rho_t = T - t_0$, $\rho_\sigma = \sigma_0/\sqrt{2}$.

Thus, the radius of convergence is unbounded for Taylor series expansions in the interest rate or dividend yield. For expansions in the stock price, Estrella (1995) proved that the radius of convergence is the stock price, so

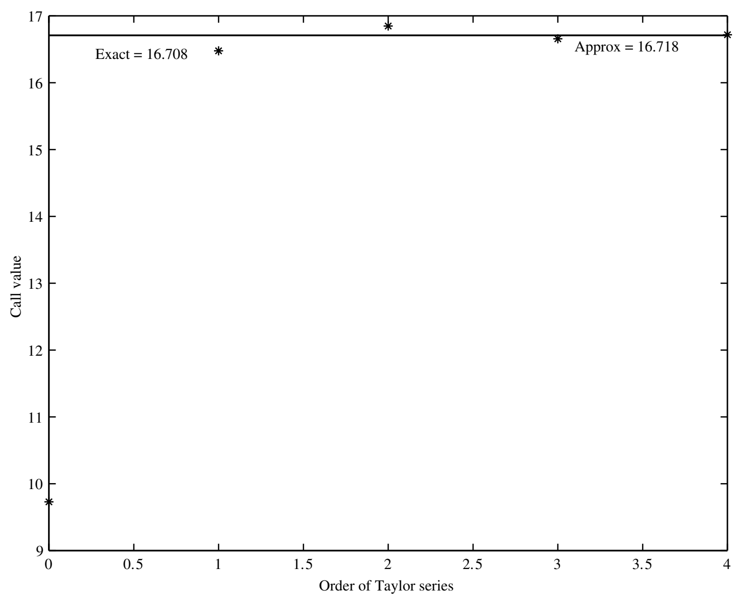

FIGURE 1. Taylor series expansion of European call value in all variables.

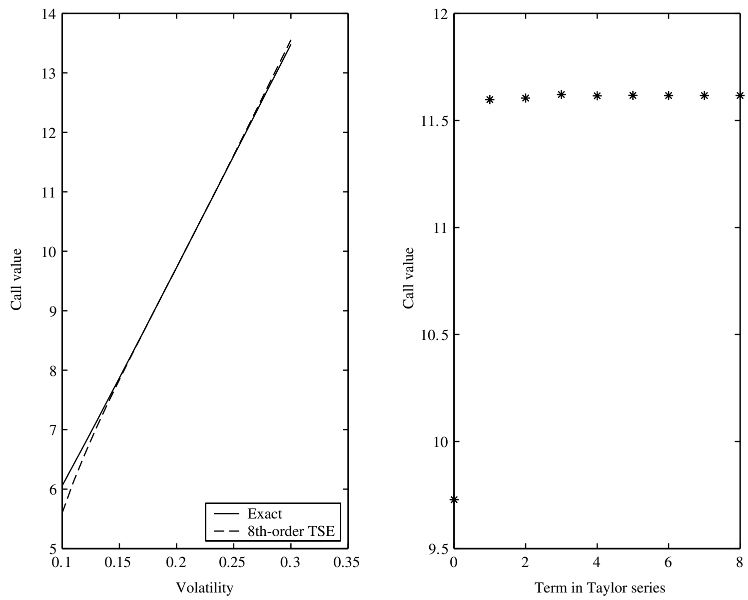

FIGURE 2. Taylor series convergence of European call value in volatility.

that Taylor series should not be used if the stock price is increased by a factor of 2 or more. Similarly, for expansions in the time variable t, a similar analysis shows that the radius of convergence is the tenor, so that Taylor series should not be used to move backwards in calendar time by more than the tenor. For expansions in volatility, the radius of convergence is $\sigma/\sqrt{2}$, so that Taylor

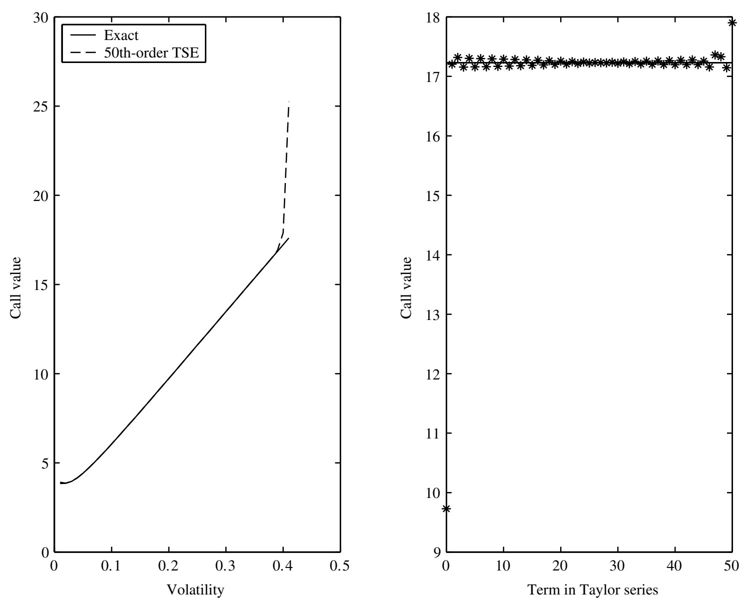

FIGURE 3. Taylor series divergence of European call value in volatility.

series should not be used if volatility is raised or lowered by more than about 70%. Since implied volatilities have been known to rise by more than this amount following a crash, it is worth exploring the Taylor series expansion in volatility when this condition is violated. Figure 3 shows such an expansion when volatility doubles from .2 to .4, holding (q, r, t, S, K, T) constant at $(.02, .06, 0, 100, 100, 1)$. Surprisingly, the expansion appears to work until about the 40th term, when it begins to diverge. This example shows the danger of increasing the order of a truncated Taylor series in an attempt to improve accuracy when convergence is not guaranteed.

7. SUMMARY AND EXTENSIONS

For path-independent claims in the BMS model, delta, gamma, speed, and higher-order price derivatives can all be interpreted as the values of certain quantoed contingent claims. This interpretation allows their values to be calculated as a discounted expectation. Any partial derivative with respect to q, r, t, σ, and/or S can be expressed in terms of the security's spatial derivatives. Since the latter are easily determined, Taylor series in all five variables become feasible. However, the efficacy of the truncated versions of these series depends on the magnitude of the change in the variables. For sufficiently large changes in S, t, or σ, Taylor series diverge.

The results of this paper realize their greatest practical significance when numerical methods must be employed to value a claim. The same technique used to numerically value the claim can be used to numerically determine spatial derivatives. Given numerical results for these spatial derivatives, the other derivatives can be determined analytically. Thus, computational resources should be spent accurately determining the claim's spatial derivatives, rather than attempting a coarser approximation of all the greeks.

Our results easily extend to contingent claims with intermediate payoffs, either discrete or continuous. The extension of our results to multiple state variables or to more complex stochastic processes or payoff structures should be explored. In the interests of brevity, these extensions are left for future research.

APPENDICES

1. Functional Analysis Proof of Theorem 1

In this appendix we prove the following general result for the jth-order derivative $D_s^j V(\tau, S) \equiv \partial^j V(\tau, S)/\partial S^j$ with respect to the underlying security price:

$$D_s^j V(\tau, S) = e^{[(j-1)r - jd + \frac{1}{2}(j-1)j\sigma^2]\tau} \mathsf{E}^{(j)}[f^{(j)}(S_T) \mid S_t = S] \quad (j = 0, 1, \ldots), \quad \text{for } S > 0, \tau > 0. \quad (35)$$

Define a family of linear operators by

$$\mathcal{L}_j \equiv \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - q + j\sigma^2) S \frac{\partial}{\partial S} + [(j-1)r - jq + \tfrac{1}{2}(j-1)j\sigma^2]\mathcal{I}, \quad (j = 0, 1, \ldots). \tag{36}$$

Then the general result (35) is implied by the Feynman–Kac formula (see Duffie 1988) if

$$\frac{\partial D_s^j V(\tau, S)}{\partial \tau} = \mathcal{L}_j D_s^j V(\tau, S) \quad (j = 0, 1, \ldots), \qquad S > 0, \quad \tau > 0. \tag{37}$$

Recall the Black–Scholes PDE:

$$\frac{\partial V(\tau, S)}{\partial \tau} = \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(\tau, S)}{\partial S^2} + (r - q) S \frac{\partial V(\tau, S)}{\partial S} - rV(\tau, S), \qquad S > 0, \quad \tau \in (0, T). \tag{38}$$

This PDE implies that (37) holds for $j = 0$:

$$\frac{\partial V(\tau, S)}{\partial \tau} = \mathcal{L}_0 V(\tau, S), \qquad S > 0, \quad \tau > 0. \tag{39}$$

To show that (37) holds for all j, we use induction. Thus, suppose that (37) holds for a particular j. To show that (37) also holds with j replaced by $j+1$, differentiate (37) with respect to S:

$$\begin{aligned}
\frac{\partial D_s^{j+1} V(\tau, S)}{\partial \tau} &= \mathcal{L}_j D_s^{j+1} V(\tau, S) + \frac{\partial \mathcal{L}_j}{\partial S} D_s^j V(\tau, S) \\
&= \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} D_s^{j+1} V(\tau, S) + (r - q + j\sigma^2) S \frac{\partial}{\partial S} D_s^{j+1} V(\tau, S) \\
&\quad + [(j-1)r - jq + \tfrac{1}{2}(j-1)j\sigma^2] D_s^{j+1} V(\tau, S) \\
&\quad + \sigma^2 S \frac{\partial^2 D_s^j V(\tau, S)}{\partial S^2} + (r - q + j\sigma^2) \frac{\partial D_s^j V(\tau, S)}{\partial S} \\
&= \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} D_s^{j+1} V(\tau, S) + [r - q + (j+1)\sigma^2] S \frac{\partial}{\partial S} D_s^{j+1} V(\tau, S) \\
&\quad + [jr - (j+1)q + \tfrac{1}{2} j(j+1)\sigma^2] D_s^{j+1} V(\tau, S) \\
&= \mathcal{L}_{j+1} D_s^{j+1} V(\tau, S). \quad \square
\end{aligned}$$

2. Probabilistic Proof of Theorem 1

In this appendix we derive Theorem 1 using purely probabilistic means. Recall our original assumption that the underlying security price follows geometric

Brownian motion:

$$\frac{\mathrm{d}S_t}{S_t} = \alpha_t \,\mathrm{d}t + \sigma \,\mathrm{d}B_t, \quad t \in [0, T], \tag{40}$$

where $\{B_t; t > 0\}$ is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. Let $\{\lambda_t \equiv (\alpha_t + q - r)/\sigma, t \in [0, T]\}$ denote the market price of risk process and define the risk-neutral probability measure $Q^{(0)}$, equivalent to P, by its Radon–Nikodym derivative

$$\frac{\mathrm{d}Q^{(0)}}{\mathrm{d}Q} = \exp\left\{-\int_0^T \tfrac{1}{2}\lambda_t^2 \,\mathrm{d}t - \int_0^T \lambda_t \,\mathrm{d}B_t\right\}.$$

Then, by Girsanov's theorem (see Karatzas and Shreve 1988, p. 191), $B_t^{(0)} \equiv B_t + \int_0^t \lambda_s \mathrm{d}s$, with $t \in [0, T]$, is a standard Brownian motion on $(\Omega, \mathcal{F}, Q^{(0)})$. Substituting into (40) gives

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= \alpha_t \,\mathrm{d}t + \sigma \,\mathrm{d}\left[B_t^{(0)} - \int_0^t \lambda_s \,\mathrm{d}s\right] \\ &= (r - q)\,\mathrm{d}t + \sigma \,\mathrm{d}B_t^{(0)}, \quad t \in [0, T], \end{aligned} \tag{41}$$

with solution

$$S_t = S_0 \mathrm{e}^{(r-q-\frac{1}{2}\sigma^2)t + \sigma B_t^{(0)}}, \quad t \in [0, T]. \tag{42}$$

Let $\mathsf{E}^{(0)}$ denote expectation under the risk-neutral measure $Q^{(0)}$. Then Harrison and Kreps (1979) and Harrison and Pliska (1981) show that the value at $t \in [0, T]$ of a path-independent claim depends on this expectation as follows:

$$V(\tau, S) = \mathrm{e}^{-r\tau} \mathsf{E}^{(0)}[f(S_T) \mid S_t = S], \qquad S > 0, \quad \tau \in [0, T], \tag{43}$$

where we recall that $f(S_T)$ is the final payoff of the claim.

To prove Theorem 1, we develop the following theorem, which is easily proved.[10]

THEOREM 5 *Define a process* $\xi_t^{(j)}$ *by*

$$\xi_t^{(j)} \equiv \mathrm{e}^{-\frac{1}{2}j^2\sigma^2\tau + j\sigma(B_T^{(0)} - B_t^{(0)})}, \quad j = 0, 1, \ldots, t \in (0, T). \tag{44}$$

Then, for any function $g(\cdot)$,

$$\mathsf{E}^{(0)}[g(S_T)\xi_t^{(j)}] = \mathsf{E}^{(j)}[g(S_T)]. \tag{45}$$

The operator $\mathsf{E}^{(j)}$ *indicates that the expectation of* $g(S_T)$ *is calculated as if the*

[10] Define the probability measure $Q^{(j)}$, equivalent to $Q^{(0)}$, by its Radon–Nikodym derivative $\mathrm{d}Q^{(j)}/\mathrm{d}Q^{(0)} = \xi_j$. Then, by Girsanov's theorem, $B_t^{(j)} \equiv B_t^{(0)} - j\sigma t$, with $t \in [0, T]$, is a standard Brownian motion on $(\Omega, \mathcal{F}, Q^{(j)})$. Substituting into (41) gives:

$$\frac{\mathrm{d}S_t}{S_t} = (r - q)\,\mathrm{d}t + \sigma \,\mathrm{d}[B_t^{(j)} + j\sigma t] = (r - q + j\sigma^2)\,\mathrm{d}t + \sigma \,\mathrm{d}B_t^j, \quad t \in [0, T].$$

underlying security's price process is

$$\frac{\mathrm{d}S_t}{S_t} = (r - q + j\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^{(j)}, \quad j = 0, 1, \ldots, t \in [0, T]. \tag{46}$$

If we differentiate the claim's value in (43) using the chain rule, we can express the claim's time-t delta $V_s(T - t, S_0)$ in terms of its final delta $f^{(1)}(S_T)$:

$$V_s(\tau, S) = \mathrm{e}^{-r\tau}\mathsf{E}^{(0)}[f^{(1)}(S_T)\mathrm{e}^{(r-q-\frac{1}{2}\sigma^2)\tau+\sigma(B_T^{(0)}-B_t^{(0)})} \mid S_t = S] \quad \text{from (42)} \tag{47}$$

$$= \mathrm{e}^{-q\tau}\mathsf{E}^{(0)}[f^{(1)}(S_T)\xi_t^{(1)} \mid S_t = S], \tag{48}$$

where $\xi_t^{(1)} \equiv \mathrm{e}^{-\frac{1}{2}\sigma^2\tau+\sigma(B_T^{(0)}-B_t^{(0)})}$, with $t \in [0, T]$. Similarly, if we differentiate the claim's delta in (47) using the chain rule, we can express the claim's initial gamma $V_{ss}(T, S_0)$ in terms of its final gamma $f^{(2)}(S_T)$:

$$\begin{aligned} V_{ss}(\tau, S) &= \mathrm{e}^{-r\tau}\mathsf{E}^{(0)}[f^{(2)}(S_T)\mathrm{e}^{(r-q-\frac{1}{2}\sigma^2)2\tau+2\sigma(B_T^{(0)}-B_t^{(0)})} \mid S_t = S] \\ &= \mathrm{e}^{(r-2q+\sigma^2)\tau}\mathsf{E}^{(0)}[f^{(2)}(S_T)\xi_t^{(2)} \mid S_t = S], \end{aligned} \tag{49}$$

where $\xi_t^{(2)} \equiv \mathrm{e}^{-2\sigma^2\tau+2\sigma(B_T^{(0)}-B_t^{(0)})}$, with $t \in [0, T]$. By repeated differentiation, the jth derivative $D_s^j V(\tau, S) \equiv \partial^j V(\tau, S)/\partial S^j$ of the claim's value with respect to the underlying security price is

$$\begin{aligned} D_s^j V(\tau, S) &= \mathrm{e}^{-r\tau}\mathsf{E}^{(0)}[f^{(j)}(S_T)\mathrm{e}^{(r-q-\frac{1}{2}\sigma^2)j\tau+j\sigma(B_T^{(0)}-B_t^{(0)})} \mid S_t = S] \\ &= \mathrm{e}^{[(j-1)r-jq+\frac{1}{2}(j-1)j\sigma^2]T}\mathsf{E}^{(0)}[f^{(j)}(S_T)\xi_t^{(j)} \mid S_t = S], \end{aligned} \tag{50}$$

where $\xi_t^{(j)} \equiv \mathrm{e}^{-\frac{1}{2}j^2\sigma^2\tau+j\sigma(B_T^0-B_t^{(0)})}$ for $j = 0, 1, \ldots, t \in [0, T]$.

Applying Theorem 5 with $g(S_T) = f^{(j)}(S_T)$ allows us to calculate the initial jth derivative $D_s^j V(\tau, S)$ at t by discounting the conditional expected final jth derivative $\mathsf{E}^{(j)}[f^{(j)}(S_T) \mid S_t = S]$:

$$D_s^j V(\tau, S) = \mathrm{e}^{[(j-1)r-jq+\frac{1}{2}(j-1)j\sigma^2]\tau}\mathsf{E}^{(j)}[f^{(j)}(S_T) \mid S_t = S] \quad (j = 0, 1, \ldots), \quad \text{for } S > 0, \tau \in (0, T). \tag{51}$$

The operator $\mathsf{E}^{(j)}$ indicates that the expectation of the final jth derivative $f^{(j)}(S_T)$ is calculated assuming that the terminal security price S_T is given by

$$S_T = S_0\mathrm{e}^{[r-q+(j-\frac{1}{2})\sigma^2]T+\sigma B_T^{(j)}} \quad (j = 0, 1, \ldots), \tag{52}$$

where $\{B_t^{(j)}, t \in [0, T]\}$ is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, Q^{(j)})$, with $j = 0, 1, \ldots$.

3. Operator Calculus

This appendix justifies the manipulations that led to equations (28)–(30) for the derivatives of claim value with respect to r, q, and σ. We begin by rewriting

(12) as

$$\frac{\partial U(\tau, x)}{\partial \tau} = LU(\tau, x), \qquad x \in (-\infty, \infty), \quad \tau > 0, \tag{53}$$

where L is a linear operator defined by

$$L \equiv \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} - r\mathcal{I}, \qquad \mu \equiv r - q - \tfrac{1}{2}\sigma^2.$$

Let $\phi(x)$ be the transformed initial condition

$$\phi(x) \equiv f(S). \tag{54}$$

Then, for each fixed $x \in \mathbb{R}$, the operational solution of the initial value problem (53, 54) is

$$U(\tau, x) = \exp\{\tau \cdot L\}\phi(x), \quad \tau \in (0, T), \tag{55}$$

where $\exp\{\tau \cdot L\}$ is another operator defined by

$$\exp\{\tau \cdot L\} \equiv \sum_{j=0}^{\infty} \frac{(\tau L)^j}{j!}. \tag{56}$$

To prove this result, we differentiate the proposed solution (55) with respect to τ:

$$\begin{aligned}
\frac{\partial U(\tau, x)}{\partial \tau} &= \frac{\partial \exp\{\tau \cdot L\}\phi(x)}{\partial \tau} \\
&= \lim_{\Delta\tau \downarrow 0} \frac{\exp\{(\tau + \Delta\tau) \cdot L\} - \exp\{\tau \cdot L\}}{\Delta\tau} \phi(x) \\
&= \lim_{\Delta\tau \downarrow 0} \frac{\exp\{\Delta\tau \cdot L\} - 1}{\Delta\tau} \exp\{\tau \cdot L\}\phi(x) \\
&= L \exp\{\tau \cdot L\}\phi(x) \\
&= LU(\tau, x), \quad \tau \in (0, T).
\end{aligned}$$

To verify equations (28)–(30) for the derivatives with respect to the dividend yield, riskless rate, and volatility, we express the linear operator L as the sum of three linear operators, each dependent on only one parameter:

$$L = L_q + L_r + L_\sigma,$$

where

$$L_q \equiv -q\frac{\partial}{\partial x}, \qquad L_r \equiv r\left(\frac{\partial}{\partial x} - \mathcal{I}\right), \qquad L_\sigma \equiv \tfrac{1}{2}\sigma^2\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right). \tag{57}$$

It is easily verified that these three operators commute, i.e.,

$$L_q L_r = L_r L_q, \qquad L_\sigma L_q = L_q L_\sigma, \qquad L_\sigma L_r = L_r L_\sigma.$$

Consequently, the exponential operator in (56) can be written as the composition of three new exponential operators:[11]

$$\exp\{\tau \cdot L\} = \exp\{\tau \cdot L_q\} \exp\{\tau \cdot L_r\} \exp\{\tau \cdot L_\sigma\}. \tag{58}$$

Since L_r, L_q, and L_σ commute, the three exponential operators also commute.

Define three new linear operators by differentiating each of the L operators defined in (57) with respect to its associated parameter:

$$L_q^{(1)} \equiv -\frac{\partial}{\partial x}, \qquad L_r^{(1)} \equiv \left(\frac{\partial}{\partial x} - \mathcal{I}\right), \qquad L_\sigma^{(1)} \equiv \sigma\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right).$$

Since each of these three new operators commutes with its corresponding operator in (57), each of these three new operators also commutes with its corresponding exponential operator in (58). Consequently, the derivatives of the exponential operator defined in (56) with respect to the three parameters satisfy

$$\frac{\partial}{\partial q}\exp\{\tau \cdot L\} = \tau L_q^{(1)} \exp\{\tau \cdot L\},$$
$$\frac{\partial}{\partial r}\exp\{\tau \cdot L\} = \tau L_r^{(1)} \exp\{\tau \cdot L\},$$
$$\frac{\partial}{\partial \sigma}\exp\{\tau \cdot L\} = \tau L_\sigma^{(1)} \exp\{\tau \cdot L\}.$$

The foregoing implies that the following manipulations are justified:

$$\frac{\partial V(\tau, S)}{\partial q} = \frac{\partial U(\tau, x)}{\partial q} = \frac{\partial}{\partial q}\exp\{\tau \cdot L\}\phi(x) = \tau L_q^{(1)} \exp\{\tau \cdot L\}\phi(x)$$
$$= -\tau \frac{\partial}{\partial x} U(\tau, x) = -\tau S \frac{\partial V(\tau, S)}{\partial S} \text{ from (10)},$$
$$\frac{\partial V(\tau, S)}{\partial r} = \frac{\partial U(\tau, x)}{\partial r} = \frac{\partial}{\partial r}\exp\{\tau \cdot L\}\phi(x) = \tau L_r^{(1)} \exp\{\tau \cdot L\}\phi(x)$$
$$= \tau\left(\frac{\partial}{\partial x} - \mathcal{I}\right) U(\tau, x) = -\tau\left[V(\tau, S) - S\frac{\partial V(\tau, S)}{\partial S}\right] \text{ from (8, 10)},$$
$$\frac{\partial V(\tau, S)}{\partial \sigma} = \frac{\partial U(\tau, x)}{\partial \sigma} = \frac{\partial}{\partial \sigma}\exp\{\tau \cdot L\}\phi(x) = \tau L_\sigma^{(1)} \exp\{\tau \cdot L\}\phi(x)$$
$$= \tau\sigma\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) U(\tau, x) = \tau\sigma S^2 \frac{\partial^2 V(\tau, S)}{\partial S^2} \text{ from (11)}.$$

[11] If two operators A and B commute, then $\exp(A + B) = \exp A \exp B$. To verify this assertion, define an operator $f(\lambda) \equiv \exp(\lambda A)\exp(\lambda B)$ for all $\lambda \in \mathbb{R}$. Then

$$\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda} = A\exp(\lambda A)\exp(\lambda B) + \exp(\lambda A) B \exp(\lambda B)$$
$$= A\exp(\lambda A)\exp(\lambda B) + B\exp(\lambda A)\exp(\lambda B) \quad \text{since } A \text{ and } B \text{ commute}$$
$$= (A + B) f(\lambda).$$

The solution to this ordinary differential equation in λ is $f(\lambda) = \exp[\lambda(A + B)]$. Setting $\lambda = 1$ verifies the assertion.

4. Stirling Numbers of the First and Second Kind

The Stirling numbers of the first kind satisfy the recursion

$$\mathcal{S}_1(l,i) = \begin{cases} \mathcal{S}_1(l-1,i-1) - (l-1)\mathcal{S}_1(l-1,i) & \text{for } l = 1,2,\ldots; i = 1,2,\ldots,l, \\ 0 & \text{otherwise,} \end{cases}$$

except that $\mathcal{S}_1(0,0) = 1$. The first few Stirling numbers of the first kind are given in the table below:

l/i	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	−1	1	0	0
3	0	2	−3	1	0
4	0	−6	11	−6	1

From Comtet (1974), the solution of the recursion is given by the following complicated closed-form solution:

$$\mathcal{S}_1(l,i) = \sum_{j=0}^{l-i}\sum_{h=j}^{l-i}(-1)^{j+h}\binom{h}{j}\binom{l-1+h}{l-i+h}\binom{2l-i}{l-i-h}\frac{(h-j)^{l-i+h}}{h!},$$
$$\text{for } l = 1,2,\ldots; i = 1,2,\ldots,l.$$

The Stirling numbers of the second kind satisfy the recursion

$$\mathcal{S}_2(l,i) = \begin{cases} \mathcal{S}_2(l-1,i-1) + i\mathcal{S}_2(l-1,i) & \text{for } l = 1,2,\ldots; i = 1,2,\ldots,l, \\ 0 & \text{otherwise,} \end{cases}$$

except that $\mathcal{S}_2(0,0) = 1$. The first few Stirling numbers of the second kind are given in the table below:

l/i	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	1	1	0	0
3	0	1	3	1	0
4	0	1	7	6	1

From Abramowitz and Stegun (1965), the solution of the recursion is given by the following simple closed-form solution:

$$\mathcal{S}_2(l,i) = \frac{1}{l!}\sum_{j=1}^{i}(-1)^{i-j}\binom{i}{j}j^l, \quad l = 1,2,\ldots; i = 1,2,\ldots,l.$$

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