Forward Evolution Equations for KnockOut Options*

Peter Carr[1], and Ali Hirsa[2]

[1] Bloomberg
731 Lexington Avenue
New York, NY 10022
pcarr4@bloomberg.net
[2] Caspian Capital Management, LLC
745 Fifth Avenue
New York, NY 10151
ali.hirsa@ixis-ccm.com

Summary. We derive forward partial integro differential equations (PIDEs) for up-and-out and down-and-out call options when the underlying is a jump diffusion. We assume that the jump part of the returns process is an additive process. This framework includes the variance gamma, finite moment logstable, Merton jump diffusion, Kou jump diffusion, Dupire, CEV, arcsinhnormal, displaced diffusion, and Black Scholes models as special cases.
Key words: Partial Integro Differential Equation (PIDE); Forward Equations; KnockOut Options; Jump Diffusion; Lévy Processes.

1 Introduction

Pricing and hedging derivatives consistent with the volatility smile has been a major research focus for over a decade. A breakthrough occurred in the mid-nineties with the recognition that in certain models, European option values satisified forward evolution equations in which the independent variables are the options' strike and maturity. More specifically, [12] showed that under deterministic carrying costs and a diffusion process for the underlying price, no arbitrage implies that European option prices satisfy a certain partial differential equation (PDE), now called the Dupire equation. Assuming that one could observe European option prices of all strikes and maturities, then this forward PDE can be used to explicitly determine the underlying's instantaneous volatility as a function of the underlying's price and time. Once this volatility function is known, the value function for European, American, and many exotic options can be determined by a wide array of standard methods. As this value function relates theoretical prices of these instruments to the underlying's price and time, it can also be used to determine many greeks of interest as well.

Aside from their use in determining the volatility function, forward equations also serve a second useful purpose. Once one knows the volatility function either by an explicit speci-

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fication or by a prior calibration, the forward PDE can be numerically solved to efficiently value a collection of European options of different strikes and maturities all written on the same underlying asset. Furthermore, as pointed out in [4], all the greeks of interest satisfy the same forward PDE and hence can also be efficiently determined in the same way.

Since the original development of forward equations for European options in continuous models, several extensions have been proposed. For example, Esser and Schlag [14] develop forward equations for European options written on the forward price rather than the spot price. Forward equations for European options in jump diffusion models were developed in Andersen and Andreasen [1] and extended by Andreasen and Carr [3]. It is straightforward to develop the relevant forward equations for European binary options or for European power options by differentiating or integrating the forward equation for standard European options. Buraschi and Dumas [6] develop forward equations for compound options[3]. In contrast to the PDE's determined by others, their evolution equation is an ordinary differential equation whose sole independent variable is the intermediate maturity date.

Given the close relationship between compound options and American options, it seems plausible that there might be a forward equation for American options. The development of such an equation has important practical implications since all listed options on individual stocks are American-style. The Dupire equation cannot be used to infer the volatility function from market prices of American options, nor can it be used to efficiently value a collection of American options of differing strikes and maturities.

This problem is addressed for American calls on stocks paying discrete dividends in Buraschi & Dumas [6] and it is also considered in a lattice setting in Chriss [10]. In [8], we direct our attention to the more difficult problem of pricing continuously exerciseable American puts in continuous time models. To do so, we depart from the diffusive models which characterize most of the previous research on forward equations in continuous time. To capture the smile, we assume that prices jump rather than assuming that the instantaneous volatility is a function of stock price and time. Dumas, Fleming & Whaley [11] find little empirical support for the Dupire model whereas there is a long history of empirical support for jump-diffusion models[4]. In particular, we assume that the returns on the underlying asset have stationary independent increments, or in other words that the log price is a Lévy process. Besides the [5] model, our framework includes as special cases the variance gamma (VG) model of Madan, Carr & Chang [18], the CGMY model of Carr, Geman, Madan & Yor [7], the finite moment logstable model of Carr & Wu [9], the Merton [?] and Kou [17] jump diffusion models, and the hyperbolic models of Eberlein, Keller & Prause [13]. In all of these models except Black Scholes, the existence of a jump component implies that the backward and forward equations contain an integral in addition to the usual partial derivatives. Despite the computational complications introduced by this term, we use finite differences to solve both of these fundamental partial integro differential equations (PIDE's). To illustrate that our forward PIDE is a viable alternative to the traditional backward approach, we calculate American option values in the diffusion extended VG[5] option pricing model and find very close agreement.

The approach to determining the forward equation for American options in [8] is to start with the well-known backward equation and then exploit the symmetries which essentially

[3] However, their definition of a compound option is non-standard in that the critical stock price is specified in the contract.

[4] For example, three recent papers documenting support for such models are [2], [7], and [9].

[5] For details on the application of finite differences to valuing American options in the VG model, see [16].

define Lévy processes. In the process of developing the forward equation, we also determine two hybrid equations of independent interest. The advantage of these hybrid equations over the forward equation is that they hold in greater generality. Depending on the problem at hand, these hybrid equations can also have large computational advantages over the backward or forward equations when the model has already been calibrated. In particular, the advantage of these hybrid equations over the backward equation is that they are more computationally efficient when one is interested in the variation of prices or greeks across strike or maturity at a fixed time, eg. market close.

The first of these hybrid equations has the stock price and maturity as independent variables. The numerical solution of this hybrid equation is an alternative to the backward equation in producing a spot slide, which shows how American option prices vary with the initial spot price of the underlying. If one is interested in understanding how this spot slide varies with maturity, then our hybrid equation is much more efficient than the backward equation. This hybrid equation also has important implications for path-dependent options such as cliquets whose payoff directly depends depends on the particular level reached by an intermediate stock price.

Our second hybrid equation has the strike price and calendar time as independent variables. The numerical solution of this hybrid equation is an alternative to the forward equation in producing an implied volatility smile at a fixed maturity. If one is interested in understanding how the model predicts that this smile will change over time, then our hybrid equation is much more computationally efficient than the forward equation. This second hybrid equation also allows parameters to have a term structure, whereas our forward equation does not[6]. Hence, if one needs to efficiently value a collection of American options of different strikes in the time-dependent Black-Scholes model, then it is far more efficient to solve our hybrid equation than to use the standard backward equation.

In this paper, we focus on forward evolution equations for knockout options. The model dynamics are jump-diffusion where jumps are additive in the log of the price.

The remainder of this paper is structured as follows. The next section introduces our assumptions for down-and-out calls and derive the forward PIDE for down-and-out calls. The following section introduces our setting and reviews the backward PIDE which governs up-and-out call values in this setting and then develops the forward equation for up-and-out call options. We present some numerical results afterwards and the final section suggests further research.

2 Down-and-Out Calls

2.1 Assumptions and Notations

Throughout this paper, we assume the standard model of perfect capital markets, continuous trading, and no arbitrage opportunities.

When a pure discount bond is used as numeraire, then it is well known that no arbitrage implies that there exists a probability measure $\mathbb{Q}$ under which all non-dividend paying asset prices are martingales. Under this measure we assume that a stock price S_t obeys the following stochastic differential equation:

$$dS_t = [r(t) - q(t)]S_{t-}dt + a(S_{t-}, t)dW_t + \int_{-\infty}^{\infty} S_{t-}(e^x - 1)[\mu(dx, dt) - \nu(x, t)dxdt], \quad (1)$$

[6] Note however that implied vol can have a term or strike structure in our Lévy setting.

for all $t \in [0, \bar{T}]$, where the initial stock price, $S_0 > 0$, is known, and $\bar{T}$ is some arbitrarily distant horizon. The process is Markov in itself since the coefficients of the stock price process at time t depend on the path only through S_{t-}, which is the pre-jump price at t. Thus, the dynamics are fully determined by the drift function $b(S,t) \equiv [r(t) - q(t)]S$, the (normal) volatility function $a(S,t)$, and the jump compensation function $\nu(x,t)$. The term dW_t denotes increments of a standard Brownian motion (SBM) W_t defined on the time set $[0, \bar{T}]$ and on a complete probability space $(\Omega, \mathcal{F}, Q)$. The random measure $\mu(dx, dt)$ counts the number of jumps of size x in the log price at time t. The function $\{\nu(x,t), x \in \mathbb{R}, t \in [0, \bar{T}]\}$ is used to compensate the jump process $J_t \equiv \int_0^t \int_{-\infty}^{\infty} S_{t-}(e^x - 1)\mu(dx, ds)$, so that the last term in (1) is the increment of a $\mathbb{Q}$ jump martingale. [7] Thus, each price change is the sum of the increment in a general diffusion process with proportional drift and the increment in a pure jump martingale, where the latter is an additive process in the log price. We restrict the function $a(S,t)$ so that the spot price is always nonnegative and absorbing at the origin[8]. In particular, we set:

$$a(0,t) = 0. \tag{2}$$

Hence, (1) describes a continuous-time Markov model for the spot price dynamics, which is both arbitrage-free and consistent with limited liability. Aside from the Markov property, the main restrictions inherent in (1) are the standard assumptions that interest rates, dividend yields, and the compensator do not depend on the spot price.

2.2 Analysis

Let time $t = 0$ denote the valuation date for a European down-and-out call option with strike price K, barrier $L \leq K$, initial spot $S_0 > L$, and maturity $T \geq 0$. Let $D_0^c(K,T)$ denote an initial price of the down-and-out call which is implied by the absence of arbitrage. Consider the product $e^{\int_t^T r(u)du}(S_t - K)^+$. By the Tanaka Meyer formula:

$$\begin{aligned}
(S_T - K)^+ &= e^{\int_0^T r(u)du}(S_0 - K)^+ + \int_0^T e^{\int_t^T r(u)du} 1(S_{t-} > K) dS_t \\
&+ \int_0^T e^{\int_t^T r(u)du} \left\{ \frac{a^2(S_{t-}, t)}{2} \delta(S_{t-} - K) - r(t)(S_{t-} - K)^+ \right\} dt \\
&+ \int_0^T e^{\int_t^T r(u)du} \int_{-\infty}^{\infty} \left[(S_{t-}e^x - K)^+ - (S_{t-} - K)^+ - 1(S_{t-} > K) S_{t-}(e^x - 1) \right] \mu(dx, dt),
\end{aligned} \tag{3}$$

where $\delta(\cdot)$ denotes a *Dirac* delta function[9] and $\mu_t(dx, dt)$ denotes the integer valued counting measure.

[7] The function $\nu(x,t)$ must have the following properties:

1. $\nu(0,t) = 0$,
2. $\int_{-\infty}^{\infty} (x^2 \wedge 1)\nu(x,t)dx < \infty, \quad t \in [0, \bar{T}]$.

[8] A sufficient condition for keeping the stock price away from the origin is to bound the lognormal volatility.

[9] The Dirac delta function is a generalized function characterized by two properties:

1. $\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$
2. $\int_{-\infty}^{\infty} \delta(x)dx = 1$.

.

Multiplying by $e^{-\int_0^T r(u)du}1(\tau_L > T)$ and taking expectations on both sides under an equivalent martingale measure $\mathbb{Q}$, we have: $D_0^c(K,T)$

$$
\begin{aligned}
D_0^c(K,T) &= (S_0-K)^+\mathbf{E}_0^{\mathbf{Q}}1(\tau_L > T) + \int_0^T e^{-\int_0^t r(u)du}\mathbf{E}_0^{\mathbf{Q}}\{1(\tau_L > T)1(S_{t-} > K)[r(t)-q(t)]S_{t-}\}dt \\
&+ \int_0^T e^{-\int_0^t r(u)du}\left\{\frac{a^2(K,t)}{2}\mathbf{E}_0^{\mathbf{Q}}[1(\tau_L > T)\delta(S_{t-}-K)] - r(t)\mathbf{E}_0^{\mathbf{Q}}[1(\tau_L > T)(S_{t-}-K)^+]\right\}dt \\
&+ \int_0^T e^{-\int_0^t r(u)du}\mathbf{E}_0^{\mathbf{Q}}1(\tau_L > T)\int_{-\infty}^{\infty}\left[(S_te^x-K)^+ - (S_t-K)^+ - 1(S_t > K)S_t(e^x-1)\right]\nu(x,t)dxdt.
\end{aligned}
$$

Differentiating w.r.t. T implies:

$$
\begin{aligned}
\frac{\partial}{\partial T}D_0^c(K,T) &= -e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\{\delta(\tau_L - T)(S_T-K)^+\} \\
&+ e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\{1(\tau_L > T)1(S_{T-} > K)[r(T)-q(T)]S_{T-}\} \\
&+ \frac{a^2(K,T)}{2}e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}[1(\tau_L > T)\delta(S_{T-}-K)] \\
&- r(T)e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}[1(\tau_L > T)(S_{T-}-K)^+] + e^{-\int_0^T r(u)du}E_0^{\mathbb{Q}}\Big\{1(\tau_L > T) \qquad (4) \\
&\int_{-\infty}^{\infty}\left[(S_{T-}e^x-K)^+ - (S_{T-}-K)^+ - 1(S_{T-} > K)S_{T-}(e^x-1)\right]\nu(x,T)dx\Big\}.
\end{aligned}
$$

The payoff in the first term vanishes. Subtracting and adding

$$
e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\{1(\tau_L > T)[r(T)-q(T)]K1(S_{T-} > K)\}
$$

to the second term on the RHS gives:

$$
\begin{aligned}
\frac{\partial}{\partial T}D_0^c(K,T) &= e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\{1(\tau_L > T)1(S_{T-} > K)[r(T)-q(T)](S_{T-}-K)\} \\
&+ e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\{1(\tau_L > T)[r(T)-q(T)]K1(S_{T-} > K)\} + \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}D_0^c(K,T) \\
&- r(T)D_0^c(K,T) + e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\Big\{1(\tau_L > T) \\
&\int_{-\infty}^{\infty}\left[e^x(S_{T-}-Ke^{-x})^+ - 1(S_{T-} > K)(S_{T-}-K+S_{T-}e^x-S_{T-})\right]\nu(x,T)dx\Big\} \\
&= [r(T)-q(T)]D_0^c(K,T) - [r(T)-q(T)]K\frac{\partial}{\partial K}D_0^c(K,T) \\
&+ \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}D_0^c(K,T) - r(T)D_0^c(K,T) + e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\Big\{1(\tau_L > T) \\
&\int_{-\infty}^{\infty}e^x\left[(S_{T-}-Ke^{-x})^+ - 1(S_{T-} > K)(S_{T-}-Ke^{-x}+K-K)\right]\nu(x,T)dx\Big\} \\
&= -q(T)D_0^c(K,T)
\end{aligned}
$$

$$- [r(T) - q(T)]K\frac{\partial}{\partial K}D_0^c(K,T) + \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}D_0^c(K,T) + e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\Bigg\{1(\tau_L > T)$$
$$\int_{-\infty}^{\infty}\left[(S_{T-} - Ke^{-x})^+ - (S_{T-} - K)^+ - \frac{\partial}{\partial K}(S_{T-} - K)^+ K(e^{-x} - 1)\right]e^x\nu(x,T)dx\Bigg\}$$
$$= -q(T)D_0^c(K,T) - [r(T) - q(T)]K\frac{\partial}{\partial K}D_0^c(K,T) + \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}D_0^c(K,T)$$
$$+ \int_{-\infty}^{\infty}\left[D_0^c(Ke^{-x},T) - D_0^c(K,T) - \frac{\partial}{\partial K}D_0^c(K,T)K(e^{-x} - 1)\right]e^x\nu(x,T)dx. \quad (5)$$

This PIDE holds on the domain $K \geq L, T \in [0, \bar{T}]$.

For a down-and-out call, the initial condition is:

$$D_0^c(K,0) = (S_0 - K)^+, \qquad K \geq L. \quad (6)$$

Since a down-and-out call behaves like a standard call as its strike approaches infinity, we have:

$$\lim_{K\uparrow\infty} D_0^c(K,T) = \lim_{K\uparrow\infty}\frac{\partial}{\partial K}D_0^c(K,T) = \lim_{K\uparrow\infty}\frac{\partial^2}{\partial K^2}D_0^c(K,T) = 0, \quad T \in [0, \bar{T}]. \quad (7)$$

For a lower boundary condition, we note that a down-and-out call on a stock with the dynamics in (1) has the same value prior to knocking out as a down-and-out call on a stock which absorbs at L. The second derivative of this latter call gives the r-discounted risk-neutral probability density for the event that the stock price has survived to at least T and is in the interval $(K, K + dK)$. Now it is well known that the appropriate boundary condition for an absorbing process is that this PDF vanishes on the boundary. Hence:

$$\lim_{K\downarrow L}\frac{\partial^2}{\partial K^2}D_0^c(K,T) = 0, \quad T \in [0, \bar{T}]. \quad (8)$$

Evaluating (5) at $K = L$ and substituting in (8) implies:

$$\frac{\partial D_0^c(L,T)}{\partial T} = \int_{-\infty}^{\infty}\left[D_0^c(Le^{-x},T) - D_0^c(L,T) - \frac{\partial}{\partial K}D_0^c(L,T)L(e^{-x} - 1)\right]e^x\nu(x,T)dx$$
$$- [r(T) - q(T)]L\frac{\partial}{\partial K}D_0^c(L,T) - q(T)D_0^c(K,T), \quad T \in [0, \bar{T}]. \quad (9)$$

This is a Robin condition as it involves the value and both its first partial derivatives along the boundary.

3 Up-and-Out Calls

3.1 The Backward Boundary Value Problem

Consider an up-and-out European call on a stock with a fixed maturity date $T \in [0, \bar{T}]$. At the first time that the stock price crosses an upper barrier H, the call knocks out. If the upper barrier is not touched prior to T, the call matures and pays $(S_T - K)^+$ at T, where $K \in [0, H)$ is the strike price.

We let τ_H denote the first passage time to H. We adopt the usual convention of setting this first passage time to infinity if the barrier is never hit. For $t \geq \tau$, the call has knocked out and is defined to be worthless at t. If $t < \tau$, then the stock price S_t must be in the continuation region $\mathcal{C} \equiv (S,t) \in (0,H) \times [0,T)$. While the call is alive, its value is given by a function, denoted $U(S,t)$, mapping $\mathcal{C}$ into the real line. In the interior of the continuation region, the partial derivatives, $\frac{\partial U}{\partial t}$, $\frac{\partial U}{\partial S}$, and $\frac{\partial^2 U}{\partial S^2}$ all exist as classical functions. No arbitrage implies that the up-and-out call value function $U(S,t)$ satisfies the following deterministic partial integro differential equation (PIDE) in the continuation region $\mathcal{C}$, i.e.

$$\int_{-\infty}^{\infty} \left[U(Se^x,t) - U(S,t) - \frac{\partial}{\partial S} U(S,t) S(e^x - 1) \right] \nu(x,t) dx + \frac{a^2(S,t)}{2} \frac{\partial^2 U(S,t)}{\partial S^2}$$
$$+ [r(t) - q(t)] S \frac{\partial U(S,t)}{\partial S} - r(t) U(S,t) + \frac{\partial U(S,t)}{\partial t} = 0, \text{ for } (S,t) \in \mathcal{C}. \quad (10)$$

A fortiori, the up-and-out call value function $U(S,t)$ solves a backward boundary value problem (BVP), consisting of the backward PIDE (10) subject to the following boundary conditions:

$$U(S,T) = (S-K)^+, \; S \in [0,H] \quad (11)$$
$$\lim_{S \downarrow 0} U(S,t) = 0, \qquad t \in [0,T] \quad (12)$$
$$\lim_{S \uparrow H} U(S,t) = 0, \qquad t \in [0,T]. \quad (13)$$

Equation (11) states that the up-and-out call is worth its intrinsic value at expiration. The *value matching conditions* (12) and (13) shows that at each $t \in [0,T)$, the up-and-out call's value tends to zero as the stock price approaches the origin or the barrier. For each t, the up-and-out call value is not in general differentiable in S at H.

Partial derivatives or integrals of the up-and-out call value with respect to K or T also satisfy a backward BVP. In particular, the second partial derivative of the up-and-out call value with respect to K, denoted $U_{kk}(S,t)$, solves the same PIDE as $U(S,t)$ in the continuation region $\mathcal{C}$, i.e.

$$\int_{-\infty}^{\infty} \left[U_{kk}(Se^x,t) - U_{kk}(S,t) - \frac{\partial}{\partial S} U_{kk}(S,t) S(e^x - 1) \right] \nu(x,t) dx + \frac{a^2(S,t)}{2} \frac{\partial^2 U_{kk}(S,t)}{\partial S^2}$$
$$+ [r(t) - q(t)] S \frac{\partial U_{kk}(S,t)}{\partial S} - r(t) U_{kk}(S,t) + \frac{\partial U_{kk}(S,t)}{\partial t} = 0, \text{ for } (S,t) \in \mathcal{C}. \quad (14)$$

The partial derivative $U_{kk}(S,t)$ is subject to the following boundary conditions:

$$U_{kk}(S,T) = \delta(S-K), \; S \in [0,H] \quad (15)$$
$$\lim_{S \downarrow 0} U_{kk}(S,t) = 0, \qquad t \in [0,T] \quad (16)$$
$$\lim_{S \uparrow H} U_{kk}(S,t) = 0, \qquad t \in [0,T]. \quad (17)$$

3.2 Forward Propagation of Up-and-Out Call Values

Until now, we have been thinking of K and T as constants. In this section, we will be varying K and T, which will induce variation in the up-and-out call value. We will also be holding S and t constant at $S_0 \in (0, H)$ and 0 respectively. We let $u(K,T)$ denote the function relating the up-and-out call value to the strike price K and the maturity T when $(S,t) = (S_0, 0)$. While we are interested in u for all $T \in [0, \bar{T}]$, we are only interested in u for all real $K \in [0, H]$.

Let $u_{kk}(K,T) \equiv U_{kk}(S_0, 0)$ be the function emphasizing the dependence of the second strike derivative of the up-and-out call value on K and T. The appendix shows that the adjoint of the backward PIDE (14) governing u_{kk} is:

$$\int_{-\infty}^{\infty} \left[e^{-x} u_{kk}(Ke^{-x}, T) + (e^x - 2) u_{kk}(K,T) + \frac{\partial}{\partial K} u_{kk}(K,T) K (e^x - 1) \right] \nu(x,T) dx \tag{18}$$
$$+ \frac{\partial^2}{\partial K^2} \left[\frac{a^2(K,T)}{2} u_{kk}(K,T) \right] - \frac{\partial}{\partial K} \{ [r(T) - q(T)] K u_{kk}(K,T) \} - r(T) u_{kk}(K,T) - \frac{\partial u_{kk}(K,T)}{\partial T} = 0,$$

for $K \in [0, H], T \in [0, \bar{T}]$.

Note that:

$$\begin{aligned} & \tfrac{\partial^2}{\partial K^2} \{ -[r(T) - q(T)] K v_k(K,T) - q(T) v(K,T) \} \\ = & \tfrac{\partial}{\partial K} \{ -[r(T) - q(T)] v_k(K,T) - [r(T) - q(T)] K v_{kk}(K,T) - q(T) v_k(K,T) \} \\ = & -\tfrac{\partial}{\partial K} \{ [r(T) - q(T)] K v_{kk}(K,T) \} - r(T) v_{kk}(K,T). \end{aligned} \tag{19}$$

Also note that:

$$\begin{aligned} & e^x \frac{\partial^2}{\partial K^2} \left\{ u(Ke^{-x}, T) - u(K,T) - u_k(K,T) K (e^{-x} - 1) \right\} \\ = & e^x \frac{\partial}{\partial K} \left\{ e^{-x} u_k(Ke^{-x}, T) - u_k(K,T) - u_{kk}(K,T) K (e^{-x} - 1) - u_k(K,T)(e^{-x} - 1) \right\} \\ = & \frac{\partial}{\partial K} \left\{ u_k(Ke^{-x}, T) - u_k(K,T) - u_{kk}(K,T) K (1 - e^x) \right\} \\ = & e^{-x} u_{kk}(Ke^{-x}, T) - u_{kk}(K,T) - u_{kkk}(K,T) K (1 - e^x) - u_{kk}(K,T)(1 - e^x) \\ = & e^{-x} u_{kk}(Ke^{-x}, T) + (e^x - 2) u_{kk}(K,T) + u_{kkk}(K,T) K (e^x - 1). \end{aligned} \tag{20}$$

Substituting (19) and (20) in (18) implies:

$$\frac{\partial^2}{\partial K^2} \left\{ \int_{-\infty}^{\infty} [u(Ke^{-x}, T) - u(K,T) - u_k(K,T) K (e^{-x} - 1)] e^x \nu(x,T) dx \right.$$
$$\left. + \frac{a^2(K,T)}{2} u_{kk}(K,T) - [r(T) - q(T)] K u_k(K,T) - q(T) u(K,T) - \frac{\partial u(K,T)}{\partial T} \right\} = 0, \tag{21}$$

for $K \in [0, H], T \in [0, \bar{T}]$.

Integrating on K twice implies:

$$\int_{-\infty}^{\infty} [u(Ke^{-x}, T) - u(K,T) - u_k(K,T) K (e^{-x} - 1)] e^x \nu(x,T) dx \tag{22}$$
$$+ \frac{a^2(K,T)}{2} u_{kk}(K,T) - [r(T) - q(T)] K u_k(K,T) - q(T) u(K,T) - \frac{\partial u(K,T)}{\partial T} = A(T) K + B(T),$$

for $K \in [0, H], T \in [0, \bar{T}]$, and where $A(T)$ and $B(T)$ are independent of K. The forward PIDE is solved subject to an initial condition:

$$u(K,0) = (S_0 - K)^+. \tag{23}$$

Suppose we regard the Lévy density $\nu(x,T)$ and volatility function $a(S,t)$ as given, and use (22) subject to (23) to determine $u(K,T)$. Then the solution of the inhomogeneous PIDE (22) subject to the initial condition (23) is not unique. As we don't know $A(T)$ and $B(T)$, we need two conditions just to determine the operator. For uniqueness, we *may* need two more independent boundary conditions as well. If both the origin and the upper barrier are accessible, then we will definitely need two additional independent boundary conditions to obtain uniqueness. However, if the origin is inaccessible (eg. for geometric Brownian motion), then we can uniquely determine up-and-out call values without specifying a lower boundary condition.

We may alternatively suppose that all up-and-out call values are already known from the marketplace, and that the objective is to uniquely determine the Lévy density $\nu(x,t)$ and the volatility function $a(S,t)$. Under this perspective, one needs to supplement (22) subject to (23) with only two boundary conditions in order to determine $A(T)$ and $B(T)$. If one somehow knows the Lévy density, then it is straightforward to solve the algebraic equation (22) for $a(S,t)$.

We now assume that the objective is to determine $u(K,T)$ given $\nu(x,t)$ and $a(S,t)$. Differentiating (22) with respect to K implies:

$$\begin{aligned} A(T) &= \int_{-\infty}^{\infty} [e^{-x}u_k(Ke^{-x},T) - u_k(K,T) - \frac{\partial u_k}{\partial K}(K,T)K(e^{-x}-1) - u_k(K,T)(e^{-x}-1)]e^x\nu(x,T)dx \\ &+ \frac{\partial}{\partial K}\left\{\frac{a^2(K,T)}{2}u_{kk}(K,T) - [r(T)-q(T)]Ku_k(K,T) - q(T)u(K,T) - \frac{\partial u(K,T)}{\partial T}\right\} \\ &= \int_{-\infty}^{\infty} [u_k(Ke^{-x},T) - u_k(K,T) - \frac{\partial u_k}{\partial K}(K,T)K(1-e^{-x})]\nu(x,T)dx \\ &+ \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}u_k(K,T) - \left\{[r(T)-q(T)]K - a(K,T)\frac{\partial a}{\partial S}(K,T)\right\}\frac{\partial}{\partial K}u_k(K,T) - r(T)u_k(K,T) \\ &- \frac{\partial u_k(K,T)}{\partial T}. \end{aligned} \tag{24}$$

As the LHS is invariant to K, the RHS is as well, and so we are free to determine $A(T)$ by evaluating the RHS at either $K=0$ or at $K=H$.

Once $A(T)$ is known, then from (22):

$$\begin{aligned} B(T) &= \int_{-\infty}^{\infty} [u(Ke^{-x},T) - u(K,T) - u_k(K,T)K(e^{-x}-1)]e^x\nu(x,T)dx \\ &+ \frac{a^2(K,T)}{2}u_{kk}(K,T) - [r(T)-q(T)]Ku_k(K,T) - q(T)u(K,T) - \frac{\partial u(K,T)}{\partial T} - A(T)K \end{aligned} \tag{25}$$

Once again, the LHS is invariant to K and so we are free to determine $B(T)$ by evaluating the RHS at either $K=0$ or at $K=H$.

Appealing to the Feynman Kac theorem, the solution to the backward BVP (10) to (13) can be represented as:

$$u(K,T) = e^{-\int_0^T r(u)du}\mathbb{E}^{\mathbb{Q}}[1(\tau_H > T)(S_T - K)^+], \tag{26}$$

where the expectation is conditional on the known initial stock price S_0. The stock price process used to calculate these so-called risk-neutral expectations is the one solving the

SDE (1). At $K = H$, the function u gives the value of a call whose underlying must cross a knockout barrier to finish above the strike:

$$u(H, T) = 0. \tag{27}$$

Differentiating (26) w.r.t. K implies that:

$$u_k(K, T) = -e^{-\int_0^T r(u)du} \mathbb{E}^{\mathbb{Q}}[1(\tau_H > T)1(S_T > K)]. \tag{28}$$

At $K = H$, the function u_k gives the value of a short binary call whose underlying must cross a knockout barrier to finish above the strike:

$$u_k(H, T) = 0. \tag{29}$$

Differentiating (28) w.r.t. K implies that:

$$u_{kk}(K, T) = e^{-\int_0^T r(u)du} \mathbb{E}^{\mathbb{Q}}\{1(\tau_H > T)\delta(S_T - K)\}, \tag{30}$$

where $\delta(\cdot)$ denotes the Dirac delta function. Hence, the function $u_{kk}(K, T)$ gives the $r-$discounted risk-neutral probability density for the event that the stock price process survives to T and that the time T stock price is in $(K, K + dK)$:

$$u_{kk}(K, T) = e^{-\int_0^T r(u)du} \frac{Q\{\tau_H > T, S_T \in (K, K + dK)\}}{dK}. \tag{31}$$

Hence, as the strike price K approaches the knockout barrier H, the probability of surviving beyond T goes to zero:

$$\lim_{K \uparrow H} u_{kk}(K, T) = 0. \tag{32}$$

Differentiating (31) w.r.t. K implies that the slope of the discounted survival probability in K is given by:

$$u_{kkk}(K, T) = e^{-\int_0^T r(u)du} \mathbb{E}^{\mathbb{Q}}\{1(\tau_H > T)\delta^{(1)}(S_T - K)\}, \tag{33}$$

where $\delta^{(1)}(\cdot)$ denotes the first derivative of a delta function. When evaluated at $K = H$, u_{kkk} does not appear to simplify. We will later show how the information at $K = 0$ allows us to relate u_{kkk} to the PDF for the first passage time τ_H.

Differentiating (26) w.r.t. T implies that:

$$\frac{\partial}{\partial T} u(K, T) = -r(T)u(K, T) + e^{-\int_0^T r(u)du} \mathbb{E}^{\mathbb{Q}}\{\delta(\tau_H - T)(S_T - K)^+\}. \tag{34}$$

One may interpret the second term as the value of a call whose notional is contingent on the first passage time to H being the call's maturity T. For any $H \leq K$, this term vanishes. Hence, evaluating (34) at $K = H$ implies:

$$\frac{\partial}{\partial T} u(H, T) = 0, \tag{35}$$

from (27). Differentiating (28) w.r.t. T implies that:

$$\frac{\partial}{\partial T} u_k(K, T) = -r(T)u_k(K, T) + e^{-\int_0^T r(u)du} \mathbb{E}^{\mathbb{Q}}\{\delta(\tau_H - T)1(S_T > K)\}. \tag{36}$$

The only difference between (34) and (36) is that the second term now represents the value of a binary call with contingent notional. Hence:

$$\frac{\partial^2}{\partial T \partial K} u(H,T) = 0, \tag{37}$$

from (29).

We have less in the way of additional boundary conditions as $K \downarrow 0$. Since an up-and-out call has less value than a standard call, we have an upper bound on the value at $K = 0$:

$$u(0,T) \leq S_0 e^{-\int_0^T q(u)du}. \tag{38}$$

Similarly, since an up-and-out binary call has less value than a standard binary, we have a lower bound on the absolute slope in K at $K = 0$:

$$u_k(0,T) \geq -e^{-\int_0^T r(u)du}. \tag{39}$$

As an up-and-out butterfly spread has the same or lesser value than a standard butterfly spread, we do have:

$$u_{kk}(0,T) = 0. \tag{40}$$

Similarly, as an up-and-out vertical spread of butterfly spreads has the same or lesser value than a vertical spread of butterfly spreads, we have:

$$u_{kkk}(0,T) = 0. \tag{41}$$

Evaluating (34) at $K = 0$ implies:

$$\frac{\partial}{\partial T} u(0,T) = -r(T)u(0,T) + e^{-\int_0^T r(u)du} \mathbb{E}^{\mathbb{Q}}[\delta(\tau_H - T)S_T]. \tag{42}$$

As the stock price must be worth H in order for the call holder to receive anything, (42) yields a simple expression for the discounted first passage time density:

$$\phi(H,T) \equiv e^{-\int_0^T r(u)du} \frac{Q\{\tau_H \in (T, T+dT)\}}{dT} = \frac{1}{H}\left[r(T)u(0,T) + \frac{\partial}{\partial T}u(0,T)\right]. \tag{43}$$

When $\nu(x,t)$ and $a(S,t)$ are known and $u(K,T)$ is to be determined, the RHS is not known ex-ante. However, when $\nu(x,t)$ and $a(S,t)$ are not known and when one can observe market prices of up-and-out calls with zero strikes (which are up-and-out shares with no dividends received prior to T), then the RHS is observable. Evaluating (36) at $K = 0$ implies:

$$\frac{\partial}{\partial T} u_k(0,T) = -r(T)u_k(0,T) + e^{-\int_0^T r(u)du} \mathbb{E}^{\mathbb{Q}}[\delta(\tau_H - T)]. \tag{44}$$

Hence, we have another simple expression for the discounted first passage time density:

$$\phi(H,T) = r(T)u_k(0,T) + \frac{\partial}{\partial T} u_k(0,T). \tag{45}$$

Evaluating (24) at $K = H$ and substituting in (29), (32), and (37) implies that for an up-and-out call:

$$A(T) = \int_0^\infty u_k(He^{-x}, T)\nu(x,T)dx + \frac{a^2(H,T)}{2}\frac{\partial^2}{\partial K^2}u_k(H,T). \tag{46}$$

Evaluating (25) at $K = H$ and substituting in (27),(29), (32), (35), and (46) implies that for an up-and-out call:

$$B(T) = \int_0^\infty u(He^{-x}, T)e^x\nu(x,T)dx - \left[\int_0^\infty u_k(He^{-x}, T)\nu(x,T)dx + \frac{a^2(H,T)}{2}\frac{\partial^2}{\partial K^2}u_k(H,T)\right]H. \tag{47}$$

We can alternatively try to determine $A(T)$ and $B(T)$ using boundary conditions for $K = 0$. Evaluating (24) at $K = 0$ and substituting in (2),(40), and (41) implies that for an up-and-out call:

$$A(T) = -r(T)u_k(0,T) - \frac{\partial u_k(0,T)}{\partial T} = -\phi(H,T), \tag{48}$$

from (45). Evaluating (25) at $K = 0$ and substituting in (2),(40),(41), and (48) implies that for an up-and-out call:

$$B(T) = -q(T)u(0,T) - \frac{\partial u(0,T)}{\partial T}. \tag{49}$$

Using $A(T)$ and $B(T)$ determined by (46) and (47) respectively, (22) becomes:

$$\begin{aligned}
&\int_{-\infty}^\infty [u(Ke^{-x}, T) - u(K,T) - u_k(K,T)K(e^{-x} - 1)]e^x\nu(x,T)dx \\
&+\frac{a^2(K,T)}{2}u_{kk}(K,T) - [r(T) - q(T)]Ku_k(K,T) - q(T)u(K,T) - \frac{\partial u(K,T)}{\partial T} \\
&= \left[\int_0^\infty u_k(He^{-x}, T)\nu(x,T)dx + \frac{a^2(H,T)}{2}u_{kkk}(H,T)\right](K - H) \\
&+\int_0^\infty u(He^{-x}, T)e^x\nu(x,T)dx,
\end{aligned} \tag{50}$$

for $K \in (0,H), T \in [0,\bar{T}]$ and for $H > S_0$. Recall that for an up-and-out call, the initial condition is:

$$u(K,0) = (S_0 - K)^+, \quad K \in [0,H), \tag{51}$$

and for $H > S_0$. For boundary conditions, (27),(29),(32),(40), and (41) are all available. As a result, we have more than enough independent boundary conditions to uniquely determine $u(K,T)$. Note that the forward operator is not local as it acts on the function $u(K,T)$ and its derivatives at both $K < H$ and at $K = H$.

As (46) and (48) both yield expressions for $A(T)$, it follows that:

$$\int_0^\infty u_k(He^{-x}, T)\nu(x,T)dx + \frac{a^2(H,T)}{2}\frac{\partial^2}{\partial K^2}u_k(H,T) = -\phi(H,T). \tag{52}$$

Substituting (52) in (50) implies that the forward PIDE for an up-and-out call can also be written as:

$$\begin{aligned}
\frac{\partial u(K,T)}{\partial T} = &\int_{-\infty}^\infty [u(Ke^{-x}, T) - u(K,T) - u_k(K,T)K(e^{-x} - 1)]e^x\nu(x,T)dx \\
&+\frac{a^2(K,T)}{2}u_{kk}(K,T) - [r(T) - q(T)]Ku_k(K,T) - q(T)u(K,T) \\
&-\phi(H,T)(H - K) - \int_0^\infty u(He^{-x}, T)e^x\nu(x,T)dx.
\end{aligned} \tag{53}$$

To interpret this PIDE financially, first note that if an investor buys a calendar spread of up-and-out calls, then the initial cost is given by the LHS. The first term on the RHS arises only from paths which survive to T and cross K then. It can be shown that this first term is the initial value of a path-dependent claim that pays the overshoots of the strike at T. The second term on the RHS arises only from paths which survive to T and finish at K. Consider the infinite position in the later maturing call at time $t = T$ if the option survives until then. This position will have infinite time value when $S_T = K$ and zero value otherwise. The greater is the local variance rate at $S_T = K$, the greater is this conditional time value and the more valuable is this position initially. The next two terms arise only from paths which survive to T and finish above K. They capture the additional carrying costs of stock and bond which are embedded in the time value of the later maturing call. The operator given by the first four terms on the RHS also represents the present value of benefits obtained at T when an investor buys a calendar spread of standard or down-and-out calls. In contrast, the last two terms in (53) have no counterpart for calendar spreads in standard or down-and-out calls. To interpet the last two terms financially, note that:

$$\begin{aligned}
&\phi(H,T)(H-K) + \int_0^\infty u(He^{-x},T)e^x \nu(x,T)dx \\
&= e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\delta(\tau_H - T)(H-K) + e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\int_0^\infty (S_{T-} - He^{-x})^+ e^x \nu(x,T)dx \\
&= e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\delta(\tau_H - T)1(S_T \geq H)(H-K) \\
&+ e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\int_0^\infty (S_{T-}e^x - H)^+ 1(\tau_H \geq T)e^x \nu(x,T)dx \\
&= e^{-\int_0^T r(u)du}\mathbb{E}_0^{\mathbb{Q}}\delta(\tau_H - T)(S_T - H)^+ \qquad (54)
\end{aligned}$$

Thus, the last two terms in (53) represent the discounted expected value of the payoff from a a call struck at H if the first passage time to H is T. Note that the possibility of this loss can cause the calendar spread value to be negative.

If we take the up-and-out call values as given by the market, and even supposing that we know the Lévy density $\nu(x,t)$, then solving (22) or (53) for the local volatility function $a(S,t)$ is problematic, unless we somehow know $a(H,T)$ or $\phi(H,T)$ *ex ante*. Fortunately, this problem is solved by using $A(T)$ and $B(T)$ determined by (48) and (49) instead. In this case, (22) becomes:

$$\begin{aligned}
&\int_{-\infty}^\infty [u(Ke^{-x},T) - u(K,T) - u_k(K,T)K(e^{-x}-1)]e^x \nu(x,T)dx \\
&+ \frac{a^2(K,T)}{2}u_{kk}(K,T) - [r(T)-q(T)]Ku_k(K,T) - q(T)u(K,T) - \frac{\partial u(K,T)}{\partial T} \\
&= \left[-r(T)u_k(0,T) - \frac{\partial u_k(0,T)}{\partial T}\right]K - q(T)u(0,T) - \frac{\partial u(0,T)}{\partial T}. \qquad (55)
\end{aligned}$$

When the up-and-out call values are given by the market and the Lévy density $\nu(x,t)$ is known, then solving (55) for the local volatility function $a(S,t)$ is straightforward. Note that our assumption that the origin is inaccessible was crucial for achieving this result.

If the Lévy density $\nu(x,t)$ and volatility function $a(S,t)$ are instead given, then one must try to solve (55) for the up-and-out call value function $u(K,T)$ on the domain $K \in (0,H), T \in [0,\bar{T}]$. Once again, this forward operator is not local. The PIDE (55) is also solved subject to the initial condition (51). Once again, we have the 5 boundary conditions (27),(29),(32),(40), and (41) available. As usual, we need at least one boundary condition to uniquely determine $u(K,T)$ once the operator is determined. In this specification, we also need to solve for the 4 functions $u(0,T), u_k(0,T), \frac{\partial u_k(0,T)}{\partial T}$, and $\frac{\partial u(0,T)}{\partial T}$ to determine the operator. Hence, it appears that $u(K,T)$ is not determined uniquely by the forward BVP involving (55). Fortunately, it is determined by the forward BVP involving (50) and so we are able to solve for either $a(S,t)$ or $u(K,T)$ through use of the appropriate forward BVP.

4 Numerical Examples

We employ the same methodology used in [8] and [16] to numerically solve the backward and forward PIDEs for both down-and-out and up-and-out calls.

For our numerical examples, we consider $\nu(x)dx$ to be the *Lévy density* for the VG process in the following form:

$$\nu(x) = \frac{e^{-\lambda_p x}}{\nu x} \text{ for } x > 0 \text{ and } \nu(x) = \frac{e^{-\lambda_n |x|}}{\nu |x|} \text{ for } x < 0$$

and

$$\lambda_p = \left(\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu}\right)^{\frac{1}{2}} - \frac{\theta}{\sigma^2} \qquad \lambda_n = \left(\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu}\right)^{\frac{1}{2}} + \frac{\theta}{\sigma^2}.$$

where σ, ν, and θ are VG parameters.

We consider the following local volatility surface:

$$\sigma(K,T) = 0.3e^{-T}\left(100/K\right)^{0.2}. \tag{56}$$

This local volatility surface is plotted in Figure (1).

Other parameters for our numerical experiments are: spot S_0=100, risk-free rate $r = 6\%$, dividend rate $q = 2\%$, and VG parameters $\sigma = 0.3$, $\nu = 0.25$, $\theta = -0.3$.

Numerical Results on Up-and-Out Calls

In the case of up-and-out, Up-Barrier is assumed to be $H = 140$. In the backward case, for each maturity and each strike, we solve the backward PIDE and extract the value for time 0 and spot 100 as shown in Figures (2), (3), and (4).

In Figure (2), the left figure illustrates the value for up-out-calls for 3-month maturity and strike 90 and the right figure displays the value for up-out-calls for 3-month maturity and strike 110. In Figure (3), the left figure illustrates the value for up-out-calls for 6-month maturity and strike 90, and the right figure displays the value for up-out-calls for 6-month maturity and strike 110. In Figure (4), the left figure illustrates the value for up-out-calls for 12-month maturity, and strike 90, and the right figure displays the value for up-out-calls for 12-month maturity, and strike 110. In the forward case, however, we just solve the forward PIDE once and extract the values at these maturities and strikes as shown in Figure (5) Table 1 summerizes the results for up-and-out calls from both backward and forward PIDEs.

As we expected, the premiums match pretty closely.

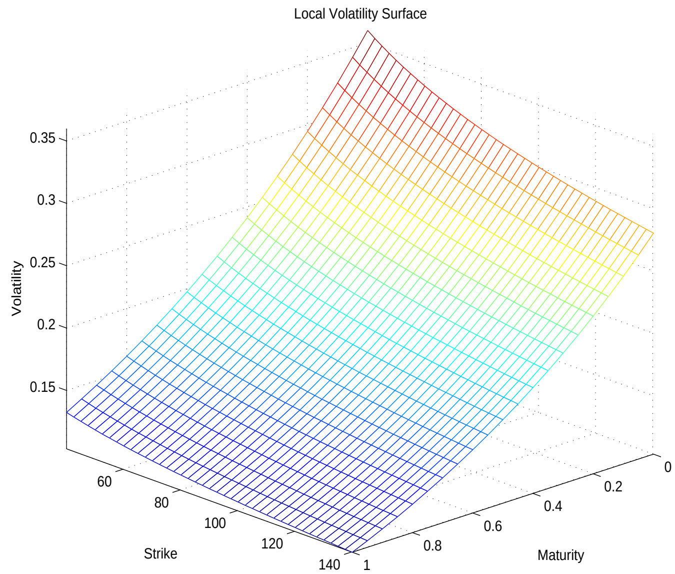

Fig. 1. Local Volatility Surface

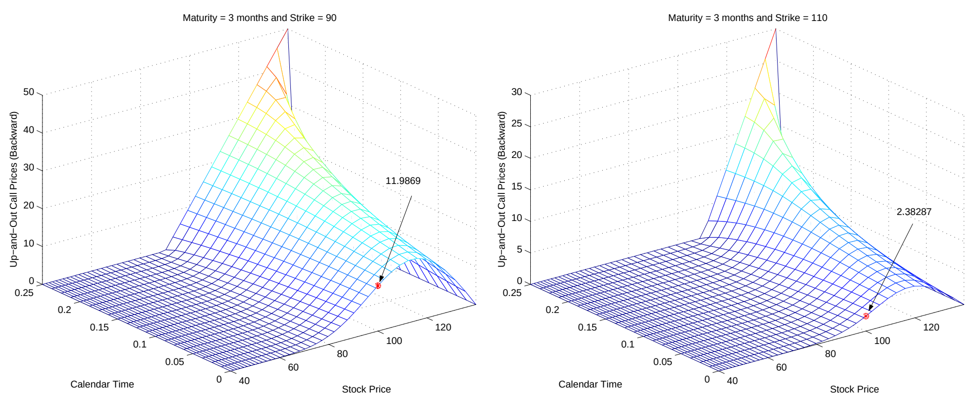

Fig. 2. At the left, the up-out-call premium for 3-month maturity and strike 90, at the right, the up-out-call premium for 3-month maturity and strike 110.

Maturity		$T_1 = 0.25$		$T_2 = 0.5$		$T_3 = 1.0$	
Barrier	Strike	**Bwd**	*Fwd*	**Bwd**	*Fwd*	**Bwd**	*Fwd*
140	90	**11.9869**	11.98901	**9.56714**	9.56918	**5.33683**	5.33786
	110	**2.38287**	2.38951	**2.32168**	2.32895	**1.28613**	1.28012

Table 1. Results for up-and-out calls for maturities: 3, 6, and 12 months, strikes: 90 and 110, and Up-Barrier $H = 140$.

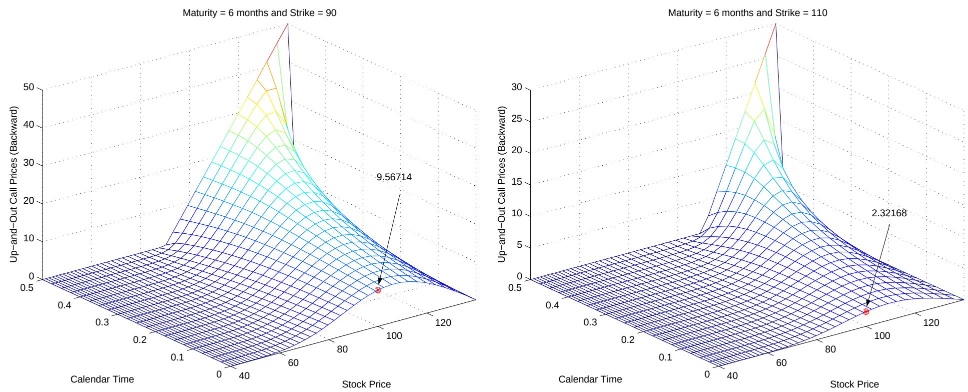

Fig. 3. At the left, the up-out-call premium for 6-month maturity and strike 90, at the right, up-out-call premium for 6-month maturity and strike 110.

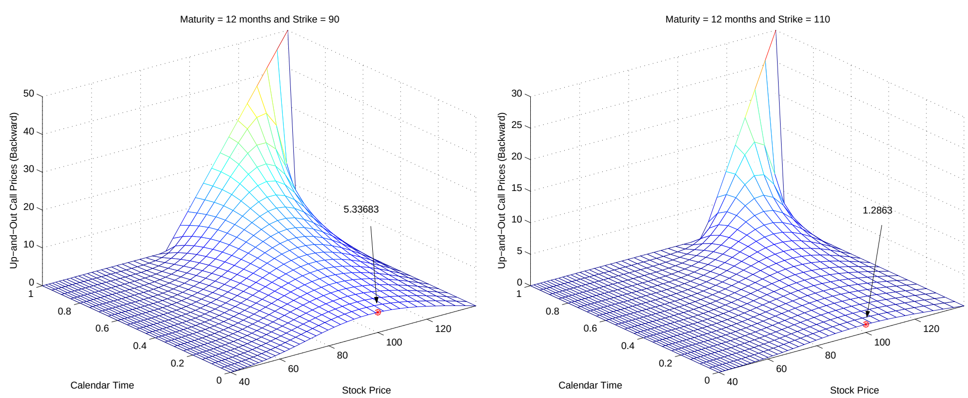

Fig. 4. At the left, the up-out-call premium for 12-month maturity and strike 90, at the right, the up-out-call premium for 12-month maturity and strike 110.

Numerical Results on Down-and-Out Calls

Table 2 illustrates the numerical results for down-and-out calls for both backward and forward PIDEs for maturities: 3, 6, and 12 months, strikes: 90 and 110. For Down-and-Out calls, Down-Barrier is assumed to be $L = 60$. As before, in the backward case, for each maturity and each strike, we solve the backward PIDE and extract the value for time 0 and spot at 100. In the forward case, we solve it once and extract the values at (K_i, T_j). As we expected, premiums are pretty identical.

5 Future Research

We would like to extend this work to other kind of barrier options such as no touches and double barrier options. One can also attempt to extend the dynamics assumption to stochastic volatility and stock jump arrival rates.

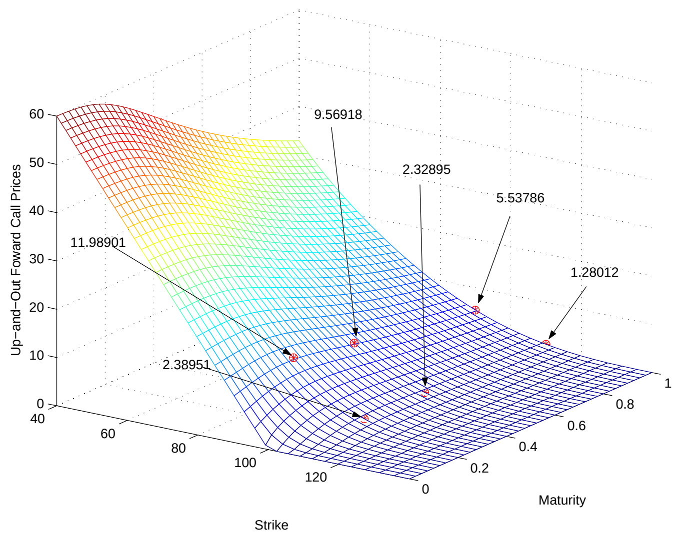

Fig. 5. Up-and-out call values for maturities: 3, 6, and 12 months and strikes: 90 and 110.

Maturity		$T_1 = 0.25$		$T_2 = 0.5$		$T_3 = 1.0$	
Barrier	Strike	**Bwd**	*Fwd*	**Bwd**	*Fwd*	**Bwd**	*Fwd*
60	90	**13.8837**	13.8798	**16.8732**	16.8801	**21.3305**	21.3345
	110	**3.57784**	3.57891	**6.99384**	6.98564	**12.1307**	12.1295

Table 2. Results for down-and-out calls for maturities: 3, 6, and 12 months, strikes: 90 and 110, and Down-Barrier $L = 60$.

A Adjoint of Backward PIDE

From Gihman and Skorohod[15], page 297:

$$\frac{\partial}{\partial T} e^{-\int_t^T r(u)du} E_t f(S_T) = -r(T) E_t e^{-\int_t^T r(u)du} f(S_T) + e^{-\int_t^T r(u)du} E_t[(\mathcal{L}_T f)(S_T)], \quad (57)$$

where:

$$\mathcal{L}_T f(S) \equiv \int_{-\infty}^{\infty} [f(Se^x) - f(S) - f'(S)S(e^x - 1)]\, \nu(x,T)dx + \frac{a^2(S,T)}{2} f''(S) + [r(T) - q(T)]Sf'(S).$$

Choose $f(S) = \delta(S - K)$. Then (57) is:

$$\frac{\partial}{\partial T} e^{-\int_t^T r(u)du} E_t \delta(S_T - K) + r(T) e^{-\int_t^T r(u)du} E_t \delta(S_T - K) = e^{-\int_t^T r(u)du} E_t[(\mathcal{L}_T \delta(S - K))(S_T)]. \quad (58)$$

The LHS of (58) is:

$$\frac{\partial}{\partial T} u_{kk}(K,T) + r(T) u_{kk}(K,T).$$

The RHS of (58) is:

$$
\begin{aligned}
&\int_0^\infty u_{kk}(L,T)\left\{\int_{-\infty}^{\infty}\left[\delta(Le^x-K)-\delta(L-K)-\delta^{(1)}(L-K)L(e^x-1)\right]\nu(x,T)dx\right.\\
&\qquad\left.+\frac{a^2(L,T)}{2}\delta^{(2)}(L-K)+[r(T)-q(T)]L\delta^{(1)}(L-K).\right\}dL\\
=&\int_{-\infty}^{\infty}\left[e^{-x}u_{kk}(Ke^{-x},T)-u_{kk}(K,T)+\left[\frac{\partial u_{kk}}{\partial K}(K,T)K+u_{kk}(K,T)\right](e^x-1)\right]\nu(x,T)dx\\
&+\frac{\partial^2}{\partial K^2}\left[\frac{a^2(K,T)}{2}u_{kk}(K,T)\right]-\frac{\partial}{\partial K}\left\{[r(T)-q(T)]Ku_{kk}(K,T)\right\}\\
=&\int_{-\infty}^{\infty}\left[e^{-x}u_{kk}(Ke^{-x},T)+(e^x-2)u_{kk}(K,T)+\frac{\partial u_{kk}}{\partial K}(K,T)K(e^x-1)\right]\nu(x,T)dx\\
&+\frac{\partial^2}{\partial K^2}\left[\frac{a^2(K,T)}{2}u_{kk}(K,T)\right]-\frac{\partial}{\partial K}\left\{[r(T)-q(T)]Ku_{kk}(K,T)\right\}. \qquad (59)
\end{aligned}
$$

Thus, the adjoint is:

$$
\begin{aligned}
\frac{\partial}{\partial T}u_{kk}(K,T)=&\int_{-\infty}^{\infty}\left[e^{-x}u_{kk}(Ke^{-x},T)+(e^x-2)u_{kk}(K,T)+\frac{\partial u_{kk}}{\partial K}(K,T)K(e^x-1)\right]\nu(x,T)dx\\
&+\frac{\partial^2}{\partial K^2}\left[\frac{a^2(K,T)}{2}u_{kk}(K,T)\right]\\
&-\frac{\partial}{\partial K}\left\{[r(T)-q(T)]Ku_{kk}(K,T)\right\}-r(T)u_{kk}(K,T). \qquad (60)
\end{aligned}
$$

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