ON DIMENSIONALLY CORRECT POWER LAW SCALING
EXPRESSIONS FOR L MODE CONFINEMENT

K.S. Riedel
Courant Institute of Mathematical Sciences
New York University
New York, New York 10012

Abstract

Confinement scalings of divertor and radiofrequency heated discharges are shown to differ significantly from the standard neutral beam heated limiter scaling. The random coefficient two stage regression algorithm is applied to a neutral beam heated limiter subset of the ITER L mode database as well as a combined dataset. We find a scaling similar to Goldston scaling for the NB limiter dataset and a scaling similar to ITER89P for the combined dataset. Various missing value algorithms are examined for the missing $B_t$ scalings. We assume that global confinement can be approximately described a power law scaling. After the second stage, the constraint of collisional Maxwell Vlasov similarity is tested and imposed. When the constraint of collisional Maxwell Vlasov similarity is imposed, the C.I.T. uncertainty is significantly reduced while the I.T.E.R. uncertainty is slightly reduced.
Global scaling expressions are widely used to analyse, interpolate, and extrapolate tokamak performance\(^1\)–\(^{13}\). Initial efforts concentrated on applying simple ordinary least squares regression using the dimensional, “engineering”, variables. Recent research has concentrated on dimensionless scalings and on incorporating the tokamak to tokamak variation into the regression analysis. In this article, we apply the random coefficient (R.C.) two step regression procedure of Refs. [4,5] while requiring the resulting expression to be dimensionless.

In Ref. [5], we showed with S. Kaye that tokamak to tokamak variation accounts for over 90% of the total variance of the scalings. To model this tokamak to tokamak variation, we treat the scaling differences between devices as random variables. This probabilistic treatment is correct when the tokamak to tokamak differences are due to many small factors. If, however, this tokamak to tokamak variation is attributable to one or more important factors such as wall material or limiter/divertor configuration, statistics is of little help in analyzing confinement.

We begin by estimating a dimensional scaling expression using the random coefficient two step regression procedure of Refs. [4,5]. The precise algorithm is discussed in detail in Refs. [4,5]. We briefly summarise the method.

First, for each tokamak, a scaling and covariance is estimated in \(I_p, B_t, \pi\) and \(P\). We calculate the empirical mean and covariance of these within tokamak scalings using the Swamy random coefficient weighting procedure. Second, the mean confinement time of each tokamak is corrected for the within tokamak scalings. The scalings with \(R/a, \kappa\) and \(R\) are estimated by
regressing the corrected mean energy times of the tokamaks. The error, $\Sigma_{RC}$ in our estimate, $\tilde{\beta}_{RC}$, of the scaling vector is given by Eq. (18a) of Ref. 4.

II. Improved Neutral Beam Limiter Confinement Scaling

In the random coefficient model, we assume that all confinement difference are random and not systematical. Since our initial statistical analysis, in collaboration with S. Kaye, we have realised that the variation in the within tokamak scalings is significantly less with only neutral beam heated (N.B.) limiter discharges in comparison to divertor discharges and radiofrequency heated (R.F.) discharges. As noted in Refs. [11,12], the overall tendency is that divertor and radiofrequency heated discharges tend to have somewhat stronger density scalings and somewhat weaker current dependencies. Table 1 gives the database summaries for the various types of discharges. In Table 2, we present the within tokamak scalings for various limiter/divertor configurations and heating types. The modified lower x point configuration of JT-60 is denoted by JTLX and is treated as a separate device.

A second feature is that the data scatter and the variation in scalings tends to be significantly larger when divertor discharges and R.F. discharges from different tokamaks are compared. This larger variability may be an artifact of that the ITER L mode database contains relatively few R.F. and divertor discharges. Furthermore, the ITER database was developed when neutral beam technology had already matured while R.F. heating was still

---

1Throughout this article, we describe the plasma current, $I_p$, in units of MAmperes, the toroidal magnetic field, $B_t$ in units of Teslas, the total heating power $P$ in MWatts, and the line averaged plasma density in $10^{19}$ particles per cubic meter.
largely under scientific study.

The R.F. and divertor data consists almost exclusively of Japanese and Soviet tokamaks. We note that the N.B limiter scalings of JT-60 and JFT-2M differ significantly from the typical N.B. limiter scaling. In fact, the difference between the R.F. and divertor scalings and the N.B. limiter scalings on JT-60 and JFT-2M appear to be less than the departure of the JT-60 and JFT-2M N.B. limiter scalings from the norm. Nevertheless, the JET divertor scaling collaborates the observed tendency of stronger density and weaker current scalings.

From Table 1, it is clear that the present database has insufficient divertor or R.F. data to perform a separate analysis. Thus we perform an analysis of only N.B. limiter data and an analysis of a combined dataset containing divertor and R.F. discharges as well. The results of the combined analysis depend on the existing mixture of datapoints and will systematically vary as more R.F. or divertor discharges are added.

We begin with an ordinary least square regression analysis of the 1346 datapoint combined database:

\[
\tau E M^{-1/2} = 0.0351 \left( \frac{R/a}{3.62} \right)^{-0.42} \left( \frac{R}{1.83} \right)^{1.60} \left( \frac{\kappa}{1.17} \right)^{0.59} \left( \frac{I_p}{0.606} \right)^{0.79} \left( \frac{B_t}{2.217} \right)^{1.13} \left( \frac{\eta}{3.947} \right)^{1.10} \left( \frac{P}{3.593} \right)^{-0.47}.
\]

In contrast, our restricted N.B limiter database has a ordinary least squares scaling of
\[ \tau_E M^{-1/2} = 0.383 \left( \frac{R/a}{3.34} \right)^{0.39} \left( \frac{R}{1.84} \right)^{1.38} \left( \frac{\kappa}{1.134} \right)^{0.61} \left( \frac{I_p}{7.005} \right)^{0.98} \left( \frac{B_t}{2.138} \right)^{1.13} \left( \frac{\pi}{4.58} \right)^{0.00} \left( \frac{P}{4.09} \right)^{-0.55}. \]

We note that the aspect ratio and size scalings are significantly modified when the combined database is used. In particular, it is difficult to believe that there is such a strong difference in aspect ratio scalings for different discharge types. We note that the major difference between the ITER89P scaling and the Goldston or Riedel-Kaye scalings is that the latter scalings are derived almost exclusively on neutral beam limiter data while the ITER89P scaling attempts to describe the combined dataset.

Our neutral beam heated limiter discharge analysis is based on a 705 datapoint subset of ITER L mode database. Our dataset consists of one small tokamak, ISX-B, three moderate size tokamaks, ASDEX, DIII and PDX and three large tokamaks, JET, JT-60 and TFTR. We restrict our analysis to discharges with \( q_{shaf} \leq 6 \). We assume that the isotope enhancement factor is \( M^{1/2} \). The \( \kappa \) scaling is treated as a between tokamak variable, instead of being determined by the \( \kappa \) scalings in DIII and ISXB.

Since our previous L mode scaling analysis \(^5\), we have added several new restrictions on the data selection procedure. First we restrict to limiter discharges since divertor discharges often have somewhat different scaling characteristics. With this restriction, the number of JET discharges is reduced from 149 to 93 and the number of JT-60 discharges is reduced from 199 to 172. The JET confinement scaling is thereby modified from \( \tau_E \sim I_p^{0.81 \pm 0.06} B_t^{0.56 \pm 0.05} \pi^{0.08 \pm 0.04} P^{-0.65 \pm 0.02} \) to \( \tau_E \sim I_p^{0.90 \pm 0.07} B_t^{0.44 \pm 0.05} \pi^{-0.05 \pm 0.04} P^{-0.57 \pm 0.03} \). Sim-
ilarly, the JT-60 scaling is modified from $\tau_E \sim I_p^{68 \pm 0.02} \pi^{18 \pm 0.02} P^{-66 \pm 0.03}$ to $\tau_E \sim I_p^{80 \pm 0.02} \pi^{08 \pm 0.02} P^{-57 \pm 0.02}$.

We have eliminated five low power or low density ASDEX datapoints, which modifies the ASDEX power scaling from $\pi^{-27 \pm 13} P^{-26 \pm 0.06}$ to $\pi^{-17 \pm 0.09} P^{-32 \pm 14}$.

Finally, we have eliminated eight low current ISXB datapoints, which modifies the ISXB scaling from $I_p^{1.42 \pm 0.07}$ to $I_p^{1.30 \pm 0.12}$.

A comparison of Table 2a with Table 2 of Ref. 5 shows that the within tokamak scalings vary significantly less in the new restricted dataset. Thus these restrictions result in a more uniform dataset which better characterises normal NBI limiter discharges.

In these low power or current discharges, the parametric dependencies are somewhat different than the typical L mode scaling. This indicates that the power law approximation to the functional form of $\tau_E$ is beginning to break down. In reality, plasma transport is a complicated nonlinear function of many parameters. A power law ansatz corresponds to a selfsimilar scaling of the dominant loss mechanisms.

As discussed in Ref. 9, the log linear form can be viewed as a Taylor series expansion of the actual functional form of $\tau_E$ about the center of mass of the database. It should come as no surprise that the L mode scaling changes and then breaks down as the extrapolation exceeds the original domain of validity. Many tokamaks have observed saturation in the current scaling below $q_a$ of three. In fact, it is surprising that the L mode power law scaling works so well over the “standard” parameter subdomain of auxiliary heated tokamaks.

When the log linear functional form is applied in too large a parame-
ter subdomain, the statistical analysis treats the systematic errors from the unresolved functional form as random errors. The precise domain of application of L mode power law scalings is a crucial and as yet unresolved area in confinement physics.

III. MISSING VALUE PROCEDURES

Since three (five for the combined database) of the tokamaks, ASDEX, JT-60, TFTR, (T-10 and JFT2M for the combined database,) have no $B_t$ variation, their $B_t$ scalings can be inferred by several different missing value algorithms. The reason for using a missing value procedure is to produce a complete second stage dataset. Since our results depend somewhat on the choice of missing value algorithm, we examine four alternatives.

The first possible missing value algorithm is to simply set the $B_t$ dependence of $\tau_E$ in ASDEX, JT-60, TFTR, (T-10, and JFT2M) equal to zero. Since many experiments observe no or an extremely weak $B_t$ dependence, this simple approach is a good first approximation. The actual Riedel-Kaye scaling of Ref.5 used this algorithm. A second missing value algorithm is to replace the missing $B_t$ dependencies by the mean value of the $B_t$ scalings in DIII, JET, ISXB, PDX (and JT60LX).

In Ref. 5, it was noted that the sum of the $B_t$ and $I_p$ scalings tended to be a constant. Therefore, a third missing value algorithm was proposed but not implemented in Ref. 5. This third algorithm consists of fitting a straight line through the $B_t$ and $I_p$ scalings. The four (or five) tokamaks with $B_t$ scans were used to determine the free parameters, $c_0$ and $c_1$ in $\beta_{B_t} = c_0 + c_1 \beta_{I_p}$, where
\( \beta_B \) is the scaling with \( B_t \) and \( \beta_I \) is the scaling with \( I_p \). Then the ASDEX, JT-60, and TFTR \( B_t \) scalings were inferred (as well as T-10 and JFT2M for the combined database). All three of these procedures are interpretive in the sense that they substitute semiempirical values for the missing values.

Finally, the "projection" missing value algorithm consists of using only the principal components of the within tokamak scalings which are estimable. This projection algorithm requires essentially no apriori assumptions about the missing scalings. The projection missing value algorithm has another advantage, it can be more easily applied to cases where in one or more tokamaks, other scaling directions, i.e. principal components, have not been varied sufficiently to be determined. The disadvantage to the projection method is that the projected data may be unbalanced and therefore illconditioned.

Virtually every reasonable missing value procedure will systematically lower the estimates of the variance, because we are replacing the random component of the \( B_t \) scaling with a more deterministic procedure. Since roughly half of the tokamaks have no \( B_t \) variation, we may underestimate the \( B_t \) variance by a factor of two.

Table 3 summaries the N.B. Limiter R.C. scalings for the various missing value algorithms. The \( B_t \) coefficient varies from .06 to .20, with the strongest \( B_t \) dependence occurring when \( \beta_B \) is regressed against \( \beta_I \). Table 4 presents the same comparison for the combined heating and magnetic configuration database.

A crude measure of the relative merits of each of the missing value procedures can be obtained by comparing the residual sum of squares in the
second stage regression on the corrected centers of mass of the various tokamaks. This measure of goodness of fit is rather inaccurate, since we are fitting four free parameters to seven datapoints. A second, independent measure of the merits of each missing value procedure is the extent to which a scaling intrinsically satisfies collisional Maxwell Vlasov similarity (see Sec. IV).

In Table 3c, 4c, the second column, $\hat{\sigma}_{2stg}$, is the R.M.S.E. for the second stage regression on the mean confinement times, weighted by the square root of the number of degrees of freedom. The third and fourth columns give the predicted energy confinement time and estimated statistical uncertainties for I.T.E.R. and C.I.T.. The fifth column is the ratio of the squared dimensional component of to its variance.

We find that the projection missing value procedure has the smallest root mean squared errors (RMSE) relative to the other missing value procedures. Thus the projection procedure shows no signs of illconditioning. The projection algorithm also satisfies C.M.V. similarity to a greater extent than the three interpretive missing value algorithms. Also, the projection procedure makes the weakest assumptions on the relational dependencies of the $B_t$ scaling. Therefore we prefer the projection algorithm to the three "interpretive"

missing value algorithms. The projection missing value procedure yields the following N.B. limiter scaling:

$$\tau_E M^{-1/2} = \left( \frac{R/a}{3.34} \right)^{.28} \left( \frac{R}{1.84} \right)^{1.22} \left( \frac{\kappa}{1.134} \right)^{.55} \left( \frac{I_p}{.7005} \right)^{1.02} \left( \frac{B_t}{2.138} \right)^{.14} \left( \frac{\bar{n}}{4.58} \right)^{.01} \left( \frac{P}{4.09} \right)^{-54}. \tag{3}$$
For the combined dataset, we find that the RMSE is the smallest for the “$B_1 = 0$” missing value procedure. However, the resulting scaling has a noticeable dimensional component. The projection procedure has a R.M.S.E. comparable to the “$B_1$” mean scaling procedure and an smaller dimensional component. Thus we select the projection procedure again. The corresponding R.C. regression for the combined dataset yields:

$$\tau_{EM}^{-1/2} = 0.0346 \left( \frac{R/a}{3.62} \right)^{-0.36} \left( \frac{R}{1.83} \right)^{1.55} \left( \kappa - 1.17 \right)^{0.63} \left( \frac{I_p}{0.606} \right)^{0.86} \left( \frac{B_1}{2.217} \right)^{1.18} \left( \frac{\pi}{3.947} \right)^{1.15} \left( \frac{P}{3.593} \right)^{-0.525}.$$  

We denote the vector of scaling coefficients by $\hat{\mathbf{\beta}}_{RC}$. Table 5 gives the 8x8 covariance matrices, $\Sigma_{RC}$, for our two-step regression vector $\hat{\mathbf{\beta}}_{RC}$ as derived in eqn. 18 of [4]. In evaluating $\Sigma_{RC}$, we include the small discharge to discharge variation term which was neglected in Ref. [5].

To evaluate the statistical uncertainty in the predicted energy confinement for a given set of parameters, we transform the tokamak’s parameters to the centered logarithmic variables, $\mathbf{x}_t$, and take the interproduct with the covariance matrix of Table 1. The centered $\mathbf{x}_t$ variable is

$$\left( \ln R - 0.609 \right) , \left( \ln \frac{R}{a} - 1.206 \right) , (\ln \kappa - 0.126) , 1 ,$$

\footnote{We present our scalings centered about the database mean, thus the mean values of our database are apparent. Also if the scaling coefficients are rounded, the overall constant in the centered formulation does not need to be adjusted. The overall constant in the noncentered version should be corrected to match the overall constant of the centered formulation.}
\[(\ln I_p + .356), (\ln B_t - .760), (\ln \pi - 1.522), (\ln P - 1.409)\],

for the N.B. limiter dataset and

\[\left((\ln R - .605), \left(\ln \frac{R}{a} - 1.29\right), (\ln \kappa - .154), 1\right),\]

\[(\ln I_p + .501), (\ln B_t - .796), (\ln \pi - 1.373), (\ln P - 1.279)\],

for the combined dataset. The fourth index corresponds to the absolute constant in the scaling law.

In Tables 3c, 4c, \(\hat{\sigma}_{2stg}\), the second stage R.M.S.E., corrected for the number of degrees of freedom, is around three to five percent for the N.B. limiter dataset and between six and eight percent for the combined dataset. Thus the fit on the combined dataset is significantly worse. The R.C. variance in our model consists of two terms. First, the variance of the absolute constant is precisely equal to \(\hat{\sigma}_{2stg}\) divided by the number of tokamaks. The second term is the variance of the within tokamak scalings. For both datasets, the within tokamak scaling variance dominates the total variance estimate. Thus the principal reason why the combined regression has a larger variance is the larger within tokamak scaling differences and not the increase in R.M.S.E..

For ITER, we assume the following parameter value: \(M = 2.5, a = 2.15m, R = 6.0m, \kappa = 2.0, I_p = 22MA, B_t = 5T, \pi = 13.8 \times 10^{19}, P_{tot} = 150MW\). The resulting predicted confinement times is 2.27 sec with an uncertainty factor of 21\% for the combined dataset scaling and 1.82 sec with an uncertainty factor of 14\% for the N.B. limiter dataset scaling. Clearly the predicted confinement times only weakly depend on the choice of missing value procedure for the \(B_t\) scaling.
For CIT, we use the following parameter values: $M = 2.5$, $a = 0.65m$, $R = 2.1m$, $\kappa = 2.0$, $I_p = 11MA$, $B_t = 10T$, $\pi = 50 \times 10^{19}$, $P_{tot} = 100MW$. We predict a CIT L mode confinement time of 392 msec with an uncertainty factor of 22% for the combined dataset scaling and a L mode confinement time of 364 msec with an uncertainty factor of 24% for the N.B. limiter scaling. The predicted confinement of C.I.T. varies about 20% depending on the choice of missing value procedure. This variation is still within the error bars. Nevertheless, the differences indicate the sensitivity of extrapolation to high field devices while the present database has little $B_t$ variation at constant size.

Since the R.M.S.E. of the combined dataset is nearly four times larger than the N.B. limiter dataset, it is difficult to understand why the C.I.T. error estimate is smaller. The C.I.T. uncertainty estimate depends strongly on the $B_t$ scaling variance. As noted earlier, the missing value procedures systematically underestimate the $B_t$ scaling variance. The combined dataset has a larger ratio (5/10) of tokamaks with no $B_t$ variance than the N.B. dataset (3/7). Thus the underestimate may be larger for the combined dataset. This may partially explain the slightly smaller C.I.T. uncertainty estimate. A second reason is that the error in our R.C. estimates is sufficiently large that the errorbars of the two estimates overlap.

In the second stage regression, we are fitting the corrected mean tokamak confinement as a function of four free parameters. Usually the use of four parameters to fit seven datapoints (or even ten datapoint for the combined dataset) would be considered overfitting. The smallest principal component of the second stage regression accounts for only about one percent of the total
variance for the N.B. dataset and .02-.03 for the combined dataset. Thus we initially believed that we could eliminate the smallest principal component. However, dropping the last component raises the RMSE by a factor of up to ten. The systematic errors are usually on the order of five to ten percent. Thus changes in the goodness of fit from $2 - 3.5\%$ to $10\%$ are significant. We therefore keep all principal components in the second stage regression.

IV. COLLISIONAL MAXWELL VLASOV CONSTRAINT

When a particular class of physical phenomena are responsible for anomalous transport, the resulting scaling expression should possess the same similarity transformations as the underlying physics instability. We consider the case where the turbulent transport is well described by the collisional Maxwell Vlasov (C.M.V.) system and the ratio of the Debeye length to all other scale lengths is infinitesimally small. This system is completely prescribed by three dimensionless variables\textsuperscript{14,15}:

$$\beta \equiv \pi T_i / B_i^2, \rho_1^* \equiv (MT_i)^{1/2} / RB_i, \nu_i^* \equiv R \pi q / T_i^2$$

(5)

together with the four naturally dimensionless variables: $\kappa$, $R/a$, $q_{cyl}$ and $M$. Thus $\tau E \Omega_i$ is completely describable as a function of these seven variables. We eliminate the temperature dependence in eqn(5) in favor of $\tau E$ using $\tau_E P = \langle nT > Vol$.

To determine dimensionless scaling expressions, we assume that the $\tau E \Omega_i$ can be modelled as a \textit{log linear function of the dimensionless variables}. For log linear functions of the C.M.V. variables, $B_i \tau E$ is a function of only
\[ v_1 = \ln \left( \frac{P}{R B_t} \right), \quad v_2 = \ln \left( \frac{MP}{R B_t} \right), \quad v_3 = \ln \left( \frac{P}{R^{3/2} \pi^{1/2} B_t} \right), \]

along with the four naturally dimensionless variables: \( \kappa, R/a, q_{cyl} \) and \( M \).

We treat this hypothesised collisional Maxwell Vlasov similarity as a constrained regression. As shown in Ref. [4], for power law scalings, this linear constraint reduces to

\[ \vec{\gamma} \cdot \vec{\beta} = -\vec{e}_B \cdot \vec{\gamma} = -\gamma_B \]  \hspace{1cm} (6)

where \( \vec{\gamma} \) is orthogonal to the 7-dimensional space spanned by the dimensionless \( \vec{\alpha}_k \). In our case,

\begin{equation}
\gamma_{R/a} = 0, \quad \gamma_\kappa = 0, \quad \gamma_R = 1, \quad \gamma_{\text{const}} = 0, \quad \gamma_{I_p} = -1/4, \quad \gamma_{B_{tor}} = -5/4, \quad \gamma_P = -3/4, \quad \gamma_{\Pi} = -2. \tag{7}
\end{equation}

We note that generally the size scaling is indeterminable within a given tokamak. Thus only by comparing a number of tokamaks can we determine a size scaling and therefore examine the C.M.V. constraint. When the major and minor radius are varied within a single device, the distance to the wall is also varied. Thus it is difficult to determine if the size scaling experiments in T.F.T.R. are heavily influenced by changes in \( Z_{eff} \) as the shape is varied.

Therefore we determine a C.M.V. constrained scaling within the multiple tokamak R.C. analysis. We denote the unit vector in the \( \vec{\gamma} \) direction by \( \hat{\vec{\gamma}} \).

To determine \( \vec{\beta}_{dl} \), we minimise the restricted least squares functional:

\[ \min_{\vec{\beta}_{dl}} (\vec{\beta}_{dl} - \vec{\beta}_{RC})^\dagger \cdot \Sigma_{RC}^{-1} \cdot (\vec{\beta}_{dl} - \vec{\beta}_{RC}) + \lambda(\vec{\gamma} \cdot \vec{\beta}_{dl} + \gamma_B). \]  \hspace{1cm} (8)

The solution is

\[ \vec{\beta}_{dl} = \vec{\beta}_{RC} + \lambda \Sigma_{RC} \cdot \vec{\gamma} \]  \hspace{1cm} (9)
where \( \lambda = - (\vec{\gamma} \cdot \hat{\vec{\beta}}_{RC} + \gamma_B) / (\vec{\gamma}^t \cdot \Sigma_{RC} \cdot \vec{\gamma}). \)

This dimensionless scaling expression minimises the difference between the generalised least squares estimator of Eqs. 3,4 and any dimensionless scaling expression as measured by the \( \Sigma_{RC}^{-1} \) metric. For any given metric, \( \Sigma_{arb}^{-1} \), Eq. 9 yields the corresponding minimising dimensionless expression, where \( \Sigma_{RC} \) is replaced by \( \Sigma_{arb} \). The common practice of arbitrarily adjusting the coefficients of a dimensional scaling to make it dimensionless, results in suboptimal scalings which can differ significantly from the closest dimensionless scaling.

To test for C.M.V. similarity in log linear scalings, we assume that the statistical model of Refs. [4,5] is correct, i.e. \( \tau_E \) has a log linear scaling with the tokamak to tokamak variation being given by the R.C. model of Ref. [4]. We now test if within this model, we can impose the additional constraint on \( \vec{\beta} \) given by Eq. 6. The expected deviation from the hypothesised dimensionless scaling is

\[
E_{xp} \left[ (\vec{\beta}_{dl} - \hat{\vec{\beta}}_{RC}) \cdot \Sigma_{RC}^{-1} \cdot (\vec{\beta}_{dl} - \hat{\vec{\beta}}_{RC}) \right] = \lambda^2 \vec{\gamma}^t \cdot \Sigma_{RC} \cdot \vec{\gamma} = (\vec{\gamma} \cdot \hat{\vec{\beta}}_{RC} + \gamma_B)^2 / \vec{\gamma}^t \cdot \Sigma_{RC} \cdot \vec{\gamma}
\]

(10)

\( \hat{\vec{\beta}}_{RC} \) and \( \Sigma_{RC} \) are basically the empirical mean and variance of the scalings of seven different tokamaks and therefore can be modeled with a \( T^2 \) distribution\(^\dagger\). Since we are interested in a single fixed component, \( \vec{\gamma} \cdot \hat{\vec{\beta}}_{RC} \), the relevant test statistic is \( T^2 \equiv |\vec{\gamma} \cdot \hat{\vec{\beta}}_{RC} + \gamma_B|^2 / \gamma^t \cdot \Sigma_{RC} \cdot \vec{\gamma} \). The \( T^2 \) statistic for this component has a \( F(1, 6) \) distribution, \( F(1, 9) \) for the combined dataset.\(^\dagger\) The \( F(1, n) \) distribution is the generalisation of the \( \chi \) distribution to the case
of an empirically determined variance, i.e. the Student T distribution. If \( \vec{\gamma} \cdot \Sigma_{RC} \cdot \vec{\gamma} \) were known and not estimated, \( T^2 = 1 \) would correspond to one standard deviation and \( T^2 = 4 \) would correspond to two standard deviations. The 50\% confidence level (corresponding to the halfwidth) for the \( F(1, 6) \) distribution is \( T^2 = 0.515 \), and for the \( F(1, 9) \) distribution is \( T^2 = 0.494 \). The 95\% confidence level for the \( F(1, 6) \) distribution is \( T^2 = 5.99 \), and for the \( F(1, 9) \) distribution is \( T^2 = 5.12 \).

For the L mode dataset, with the projection missing value procedure, the test statistic, \( T^2 = |\vec{\gamma} \cdot \hat{\beta}_{RC} + \gamma_B|^2 / \vec{\gamma} \cdot \Sigma_{RC} \cdot \vec{\gamma} = 0.168 \) for the NB limiter dataset and 0.044 for the combined dataset. These \( T^2 \) values are so small that we can not only set the dimensional projection equal to zero, but also eliminate the R.C. variance in the dimensional direction from our uncertainty estimates.

The use of the \( T^2 \) distribution is only strictly valid for within scaling vectors. The precise probability distribution of between scaling vectors is almost indeterminable. Since we are not interested in the tail of the probability distribution, this influences our results only weakly.

Of course, if all the edge physics, deposition physics and radiative losses were accurately modeled the hypothesised dependence on collisional Maxwell Vlasov variables should be trivially true. Nevertheless, the power law form in C.M.V. variables would still be a crude approximation. We can interpret the "extra" dimensional variable as an auxiliary moment of the input variables. Thus we could view our statistical hypothesis as an attempt to eliminate the use of this auxiliary moment in our modeling of confinement.

The constrained scaling vector given by Eq. (6):
\[
\tau_E M^{-1/2} = 0.0383 \left( \frac{R/a}{3.34} \right)^{0.270} \left( \frac{R}{1.84} \right)^{1.277} \left( \frac{\kappa}{1.134} \right)^{0.548} \left( \frac{I_p}{0.7005} \right)^{1.009} \left( \frac{B_t}{2.138} \right)^{1.133} \left( \frac{\pi}{4.58} \right)^{0.0096} \left( \frac{P}{4.09} \right)^{-0.548 .}
\]

We give the scaling coefficients to three digits accuracy, not because of precision, but to reduce the extent which rounding error induces a violation of C.M.V. similarity. The constrained scaling for the combined database is

\[
\tau_E M^{-1/2} =
0.0346 \left( \frac{R/a}{3.62} \right)^{-0.367} \left( \frac{R}{1.83} \right)^{1.575} \left( \frac{\kappa}{1.17} \right)^{0.629} \left( \frac{I_p}{0.606} \right)^{0.851} \left( \frac{B_t}{2.217} \right)^{1.71} \left( \frac{\pi}{3.947} \right)^{1.46} \left( \frac{P}{3.593} \right)^{-0.525} .
\]

The constrained, combined dataset scaling of Eq. 12 yields a predicted I.T.E.R. confinement time of 2.27 sec ±21% and a predicted C.I.T. confinement time of .383 sec ±17%

If the dimensionless scaling expression is accepted, the variance, \( \Sigma_{dl} \), of the estimate, \( \vec{b}_{dl} \), is the projection of \( \Sigma_{RC} \) onto the dimensionless subspace, i.e. \( \Sigma_{dl} = \Sigma_{RC} - \Sigma_{RC} \hat{\gamma} \hat{\gamma} \Sigma_{RC} / (\hat{\gamma} \Sigma_{RC} \hat{\gamma}) \). If the dimensionless scaling expression is accepted, the estimated uncertainty arising from the constrained random coefficient model with the projection procedure is reduced to 15 – 20% percent for both I.T.E.R. and C.I.T.. The unaccounted for uncertainties are discussed in Ref. [5].

We note that the C.M.V. constraint reduces the estimated C.I.T. uncertainty significantly more than the estimated I.T.E.R. uncertainty. We conjecture that this arises for the following reason. Both I.T.E.R. and C.I.T. parameters have been chosen largely from physics considerations. Thus, in some sense, I.T.E.R. and C.I.T. have minimised the dimensional component
of the extrapolation subject to engineering constraints. However since the present database lacks high field tokamaks, the extrapolation to C.I.T. entails a larger dimensional extrapolation.

V. DISCUSSION

Global scaling expressions, in particular, the Goldston-Aachen scaling\textsuperscript{1}, have been successful in predicting the energy confinement in the present generation of large tokamaks. Recently, a partial consensus\textsuperscript{12} has emerged that confinement data could best be fitted and extrapolated using the ITER89P scaling\textsuperscript{11} or a Goldston-like scaling. We note that both the Goldston and Riedel-Kaye scalings were derived and optimised for N.B. discharges. The ITER89P scaling is based on a larger class of discharges.

As discussed earlier, when restricted to a single class of discharges, the R.C. model becomes a statistically justifiable, in some senses optimal, approach. However the most important use of scaling expressions is to extrapolate confinement to next generation devices. Since future ignition tokamaks will be divertor devices with $\alpha$ particle and R.F. heating, it only seems reasonable to include the R.F. and divertor discharges in the database when extrapolating to reactor relevant plasmas.

However, since the combined scaling is a weighted average over a diverse set of operating conditions, it probably reflects to many different transport processes. Thus Eq. 12 should not be compared with specific theoretical transport models. The N.B. limiter scalings tend to be more uniform and may be of use for physical understanding. We note that although the N.B.
limiter scaling is nearly uniformly observed, it probably is strongly influenced by power deposition and edge physics effects. Eq. 11 is also probably more accurate in fitting the standard N.B. limiter plasmas. Much of the nonN.B. limiter data, on which the combined scaling is based, has a mixed reputation. Thus supporters of scalings similar to Eq. 11 can reasonably argue that the combined dataset is not sufficiently reliable to modify the relatively solid N.B. scalings. Assuming the additional data is noisier, it is unclear whether the inclusion of the R.F. and divertor data results in more accurate predictions for C.I.T. and I.T.E.R.

An abstract summary of the situation is as follows. The standard N.B. limiter database subset is probably of higher quality and may differ systematically from the R.F. divertor discharges. If there were no systematic differences and the relative variances of the N.B. and R.F. data are known, the most accurate statistical estimate of the scaling is given by weighting each tokamak inversely proportional to its variance.

Since we do not know the relative variances of the data from each tokamak, we have weighted all tokamaks in our analysis equally. If the variance of the R.F. divertor data were considerably larger than the N.B. data, the equal tokamak weighting incorrectly weights the tokamaks and may even have a higher variance than the analysis which excludes this additional data!

In reality, the major danger is systematic and not random errors. The systematic differences occur from both different edge and heating physics as well as possible poorer operational conditions. One approach to the systematic physics differences would be to estimate the R.F. divertor scaling using the N.B. limiter scaling and estimated variance as a Bayesian prior.
Our analysis has not considered systematic differences and the combined analysis weighted all tokamaks equally. In reality, the consensus of the confinement community appears to be that the N.B. limiter discharges in the database should be weighted more heavily than the preliminary R.F. divertor data. This prior information can be accommodated by assigning each tokamak in the database an apriori variance $\Delta_k$ proportional to the estimated random coefficient matrix, $\Delta$, i.e. $\Delta_k \equiv \alpha_k \Delta$ where $\alpha_k$ is given.

A simple alternative to this reweighted random coefficient regression is to average the N.B. Limiter scaling of Eq. 11 with the combined scaling of Eq. 12. We recommend a simple geometric average of the two scalings be used for extrapolating to future large scale devices. The geometric average of the two scaling is the arithmetic average on the logarithmic scale and corresponds to weighting the neutral beam limiter discharges roughly a factor of two more than the R.F. divertor discharges. This geometric average of two constrained log linear scalings will automatically satisfy collisional Maxwell Vlasov similarity. This compromise scaling has the parametric representation,

$$
\tau_{E}^{comp} = 0.03686 M^{1/2} (R/a)^{-0.049} R^{1.426} \kappa^{0.588} I_p^{0.930} B_t^{0.152} \pi^{0.078} P^{-0.537}.
$$  \hspace{1cm} (13)

This geometric average of the two scalings also has the advantage that it lies midway between the Goldston and ITER89P scalings which represent the consensus of the confinement community. Since the two scalings are not independent and in fact use roughly the same data the standard formula for the variance of two independent estimates of a predicted value is not applicable! We therefore suggest that the variance of the averaged scaling
for any predicted value such as I.T.E.R. or C.I.T. be approximated by the simple average of the variances for the two predictions.

The major practical difference between the Goldston and ITER89P scalings or the N.B. limiter scaling of Eq. 11 and the combined scaling of Eq. 12 is the exponent on the aspect ratio scaling. The additional R.F. divertor data is at a higher aspect ratio than most of the N.B. limiter data. The additional data decreases the aspect ratio scaling because the new data on average attains lower values of confinement than predicted by the N.B. limiter scaling with a strong favorable aspect ratio scaling. The geometric average of the two scalings, Eqs. 11 and 12, will have virtually no aspect ratio scaling.

The Riedel-Kaye scaling of Ref. [5] is strikingly similar to the Goldston scaling. The Riedel-Kaye algorithm differs from the original Kaye-Goldston algorithm by not only correctly weighting the scalings of the various tokamaks, but also by treating $\kappa$ as a between variable. We suspect the physics of small, unoptimised $\kappa$ variations in a single tokamak is different than large variations in different devices. We speculate that the old Kaye-Goldston algorithm, modified only by treating $\kappa$ as a between variable, might yield a scaling similar to Refs. [1,5].

Although power law scaling ansatz is a crude model which lacks a physical basis, the traditional log linear scaling incorporates all the major engineering variables with a similarity type behavior. It has been our experience that the ITER L mode database is too poorly structured to allow models with more free parameters, such as offset linear scalings, to be reliably fitted. In this article, we have examined whether a model with fewer free parameters, the collisional Maxwell Vlasov model can be used to model the data. To
answer this question, we must accurately model the errors associated with the scaling. Since we have done this within the framework of our two stage RC model, we now discuss how our choice of RC model effects the analysis of the C.M.V. constraint.

By fitting a simple power law scaling to complex loss mechanisms, we make systematic errors. This effects our statistical test of collisional Maxwell Vlasov similarity in two ways. First, in our RC model, these biased errors are interpreted as random errors, which thereby increase our estimate of $\Sigma_{RC}$. Second, the extent to which confinement violates collisional Maxwell Vlasov similarity may be increased or decreased by the biased errors. Since our statistical test is the ratio of these two terms, both of which are biased upward, the overall tendency of the systematic errors is difficult to assess.

As discussed in [4], the present data is insufficient to compute the entire $\Sigma_{RC}$ matrix. Thus the between and crosscovariance are specified by our statistical model and not determined empirically. Our choice of R.C. model is the most nearly homoscedastic possible with the empirically estimated within covariance, $\Delta$. In this context, homoscedasticity means that the variance of the random variable component of the scalings has minimal parametric dependencies. If the actual $\Sigma_{RC}$ differs significantly from our nearly homoscedastic model, the real statistic for C.M.V. similarity might be significantly different than that of Sec. II.

We now discuss the advantages of our method of constraining the variables to C.M.V. similarity relative to other approaches. An extremely naive approach is to simply regress $B_t \tau_E$ versus the seven dimensionless variables. Linear regression$^9,10$ postulates that "no measurement errors occur in the
variables. Since the largest measurement errors occur in $\tau_E$, the regression needs to be formulated so that most of the errors are in the dependent variable. The naive regression using $\beta, \nu^*, \text{and } \rho^*$ is actually more poorly conditioned since the dependent and independent variables are both explicitly defined as powers of the poorly measured variable.

The next level of sophistication is to commonly termed the "power formulation" and involves explicitly using only three of the four possible combinations of $R, B_t, \bar{n}$ and $P$. This procedure would be adequate if a simple least squares, uncorrelated approach were sufficient. Unfortunately, the tokamak to tokamak variation requires a two stage R.C. regression.

In our present formulation, we have treated the individual tokamak scalings as the basic observed quantity. This corresponds to neglecting the within tokamak variation. We have also applied the within tokamak test statistic for the constraint even though the size scaling component of the constraint is in the between tokamak direction. Our previous formulation of the C.M.V. constraint treated the individual discharges as the basic observation and used the R.C. matrix to determine a general $\Sigma$ matrix. Although this earlier formulation is more general, and treated both between and within variation, it is difficult to apply in practice. The previous treatment required the use of $(X^t\hat{\Sigma}^{-1}X)^{-1}$. Since $\Sigma_{RC}$ is poorly conditioned, the estimate of $(X^t\hat{\Sigma}^{-1}X)^{-1}$, which is needed in Ref. 4, can be wildly inaccurate.

The concurrent work of Christiansen, et. al differs from this work by applying ordinary least squares analysis to the L mode database. Thus it treats the within tokamak errors while not considering the between tokamak errors.
It is possible to perform a dimensionless two stage R.C. regression using $q_{cyl}$ and the three dimensionless combinations of $R$, $B_t$, $\pi$ and $P$ as within covariates. This amounts to determining the size scaling as a within covariate and only $R/a$ and $\kappa$ as between covariates. We strongly disfavor this approach since the size variation constitutes the largest principal component and therefore the $R$ scaling is the easiest to determine of the between scalings.

In conclusion, we have treated the collisional Maxwell Vlasov similarity ansatz for power law scalings as a constraint. Because the random coefficient algorithm not only produces efficient estimates of the parametric scalings but also a covariance matrix for the errors in the scaling, we are able to test this similarity ansatz. When the constraint of collisional Maxwell Vlasov similarity is imposed, the C.I.T. uncertainty is significantly reduced while the I.T.E.R. uncertainty is slightly reduced.

For future R.F. and alpha particle heated divertor experiments, we believe that the constrained, combined dataset scaling of Eq. 12 represents the most reliable extrapolation method. We find a predicted I.T.E.R. confinement time of 2.27 sec ±21% and a predicted C.I.T. confinement time of .383 sec ±17%.

Acknowledgment

The author thanks Geoff Cordey for discussions about dimensionless scaling constraints. Section II. was influenced in part on Ref. 14, which was written in collaboration with O. Kardaun, T. Takizuka and P. Yushmanov. The author thanks F. Perkins, C. Bolton, R. Goldston, S. Kaye, and K. Lackner for repeatedly urging him to do dimensionless regressions.

This work was supported under U.S. Department of Energy Grant No.
DE-FG02-86ER53223.
REFERENCES


