Abstract

The ‘Alfvén Paradox’ is that as resistivity decreases, the discrete eigenmodes do not converge to the generalized eigenmodes of the ideal Alfvén continuum. To resolve the paradox, the $\epsilon$-pseudospectrum of the RMHD operator is considered. It is proven that for any $\epsilon$, the $\epsilon$-pseudospectrum contains the Alfvén continuum for sufficiently small resistivity. Formal $\epsilon$-pseudoeigenmodes are constructed using the formal Wentzel-Kramers-Brillouin-Jeffreys solutions, and it is shown that the entire stable half-annulus of complex frequencies with $\rho|\omega|^2 = |k \cdot B(x)|^2$ is resonant to order $\epsilon$, i.e. belongs to the $\epsilon$-pseudospectrum. The resistive eigenmodes are exponentially ill-conditioned as a basis and the condition number is proportional to $\exp(R_M^{1/2})$, where $R_M$ is the magnetic Reynolds number.

Keywords: resistive magnetohydrodynamics, pseudospectrum, non-normal operators, continuous spectrum, Alfvén waves, magnetohydrodynamic stability.

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I. INTRODUCTION

In magnetohydrodynamics (MHD), Alfvén waves are represented as continuous spectra of the linear MHD operator\(^1\text{−}^5\), where every field line oscillates with its own frequency given by \(\omega_a(x) = k \cdot B(x)\). Alfvén wave heating is based on resonant absorption by phase-mixing at the Alfvén resonance\(^3,6\). When resistivity is included in the linear MHD equations, the Alfvén continuum is replaced by a discrete set of eigenmodes\(^7\text{−}^{11}\).

One would naively expect that the normal-mode analysis of dissipative MHD would converge to the ideal spectrum in the limit of asymptotically small resistivity. As the resistivity, \(\eta\), decreases, the distance between eigenfrequencies decreases as \(\eta^{1/2}\). The resistive eigenvalues lie on specific curves in the stable frequency half-plane, and these curves are independent of resistivity for small resistivity. The resistive magnetohydrodynamics (RMHD) paradox is that the resistive eigenmodes do not converge to the ideal continuum as the resistivity becomes vanishingly small.

To resolve this paradox, we consider the \(\epsilon\)−pseudospectrum\(^12\text{−}^{17}\), a generalization of the spectrum which corresponds to approximate eigenmodes. We show that for any \(\epsilon\), the \(\epsilon\)-pseudospectrum of resistive MHD contains the continuous spectrum of ideal MHD for sufficiently small values of the resistivity, \(\eta\). Using the Wentzel-Kramers-Brillouin-Jeffreys (WKBJ) approximation\(^8\text{−}^{11}\), we show that the entire half-annulus, \(\rho |\omega|^2 = |k \cdot B(x)|^2, \text{Im}[\omega] > 0\), is contained in the \(\epsilon\)-pseudospectrum with the critical value of \(\epsilon\) required for the existence of a \(\epsilon\)-eigenmode, \(\epsilon_{\text{crit}} \sim \exp(-1/\eta^{1/2})\).

Since the resistive spectrum and the ideal spectrum are different, we examine the question: “Which spectrum is more relevant in describing the time evolution on the ideal MHD time scale?” Perturbations in ideal MHD decay algebraically due to phase mixing. If the resistive MHD eigenmodes form a complete basis, then one would expect that initial perturbations would decay exponentially. The strong damping of the resistive eigenmodes has caused authors\(^9,10\) to question the completeness of the resistive spectrum and the significance of the resistive spectrum. For a similar problem in fluid dynamics, it has been shown that the Orr-Sommerfeld equations have a complete set of eigenmodes\(^18\). Therefore, it is reasonable to believe that the resistive MHD eigenmodes are also complete. We show that the resistive eigenmodes are strongly non-orthogonal and that the condition number of the RMHD eigenfunction basis degrades exponentially with the square root of the magnetic Reynolds number. Consequently, expanding an arbitrary initial perturbation in eigenmodes gives an ill-conditioned representation of the time evolution until times of order \(O(R_M^{3/2})\), where \(R_M\) is the magnetic Reynolds number.

In Section II, the resistive MHD equations are presented. In Section III, the \(\epsilon\)-pseudospectrum is defined. In Section IV, we show that for small resistivity, the continuous spectrum of ideal MHD is contained in \(\epsilon\)-pseudospectrum of resistive MHD. In Section V, we analyze the \(\epsilon\)-pseudospectrum using the WKBJ expansion. Section VI presents our numerical results. Section VII considers representations of the initial value problem in terms of the RMHD eigenmodes. Section VIII discusses the transient growth problem\(^12\text{−}^{15},17,19\). Section IX summarizes our findings. Appendix A gives the appropriate generalizations of \(\epsilon\)-pseudospectra to the generalized
eigenvalue problem. Appendix B evaluates the WKBJ phase integral for the linear profile. Appendix C presents our finite-element discretization. In Appendix D, the WKBJ approximation for the $\epsilon$-pseudospectrum is presented. Appendix E shows that transient growth occurs in ideal MHD when the initial perturbation is tearing mode-like.

II. LINEAR MAGNETOHYDRODYNAMICS

We denote the equilibrium magnetic field by $B(x)$ and the equilibrium current by $J(x)$. We consider incompressible MHD equations linearized about a no-flow equilibrium ($V(x) \equiv 0$) with constant density, $\rho \equiv 1$:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{J} \times \mathbf{b} + (\nabla \times \mathbf{b}) \times \mathbf{B} - \nabla p$$  \hspace{1cm} (1a)

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{b})$$  \hspace{1cm} (1b)

$$\nabla \cdot \mathbf{v} = 0$$  \hspace{1cm} (1c)

$$\nabla \cdot \mathbf{b} = 0 ,$$  \hspace{1cm} (1d)

where $\mathbf{v}$, $\mathbf{b}$, and $p$ denote the flow velocity, magnetic field and pressure perturbations. We consider time dependent perturbations of the form $\mathbf{b}(x,t) = e^{\lambda t} \mathbf{b}(x)$, and define $\omega = -i\lambda$. We write the resulting eigenvalue problem symbolically as $L_\eta \mathbf{U} = \lambda \mathbf{U}$, where $\mathbf{U}$ is the state vector: $\mathbf{U} \equiv (\mathbf{v}, \mathbf{b})^T$ and $L_\eta$ is the resistive MHD operator.

In the absence of resistivity, the $\delta W$ energy principle shows that ideal MHD is a self-adjoint operator$^{1,2,5}$. (More precisely, when the perturbed magnetic field is eliminated, Eq. (1) is rewritten as $F \xi = \rho \omega^2 \xi$, where $\xi \equiv \mathbf{v}/\lambda$ and $F$ is a symmetric operator$^1$ in $\omega^2$. In Ref. 5, it is shown that $(F + \sigma I)$ is self-adjoint where $\sigma I$ is a multiple of the identity operator which makes the combined operator positive.) Note that the ideal MHD operator is self-adjoint only when the initial perturbation has the form: $\mathbf{v}(x,t=0) = \xi$, $\mathbf{b}(x,t=0) = \nabla \times (\xi \times \mathbf{B})$

Since the ideal MHD operator is normal, an initial perturbation can be represented as a sum over the discrete eigenmodes plus an integral over the generalized eigenfunctions of the continuous spectrum. The eigenfunctions have a $\frac{1}{\xi}$ singularity at the resonant point where $\omega = \pm \mathbf{k} \cdot \mathbf{B}(x)$. In contrast, linear resistive MHD is not a normal system of equations, and thus, the eigenmodes need not be orthogonal or even form a complete basis. In Sections IV and V, we show that the resistive eigenmodes are strongly non-orthogonal.

III. EPSILON-PSEUDOSPECTRUM

The spectrum of a linear operator corresponds to complex frequencies where the frequency response Green’s function is infinite, and these frequencies dominate the long time asymptotics. The finite-time evolution of a non-normal operator can
be significantly modified by frequencies where the Green’s function is large, but not infinite. Therefore, we consider a generalization of the spectrum to near-resonance, the $\epsilon$-pseudospectrum:

**Definition 1** [Ref. 12]: Let $A$ be a closed linear operator with domain, $\mathcal{D}(A)$, and let $\epsilon \geq 0$ be given. A complex number $\lambda$ is in the $\epsilon$-pseudospectrum of $A$, which we denote by $\Lambda_\epsilon(A)$, if one of the following equivalent conditions is satisfied:

(i) the smallest singular value of $A - \lambda I$ is less than or equal to $\epsilon$.

(ii) there exists $u \in \mathcal{D}(A)$ such that $\|u\|^2 = 1$ and $||(A - \lambda I)u||^2 \leq \epsilon^2$.

(iii) $\lambda \in \Lambda(A)$ or $\lambda \in \rho(A)$ and there exists $u \in \mathcal{D}(A)$ such that $\|u\|^2 = 1$ and $u^*(A - \lambda I)^{-1}u \geq 1/\epsilon^2$.

(iv) $\lambda$ is in the spectrum of $A + \epsilon E$, where the operator $E$ satisfies $\|E\| \leq 1$.

The stated definition is for finite dimensional operators. For infinite dimension problems, we need to extend these definitions by replacing $u$ with a sequence of functions, $\{u_n\}$, in the domain of $A$, i.e. require that Definition 1 hold on the closure of the domain of $A$. Thus, condition (iii) becomes $||(A - \lambda I)^{-1}|| \geq 1/\epsilon$.

We measure the size of the operator and of the perturbation in the operator norm; i.e. $\|A\| \equiv \sup_u \|Au\|/\|u\|$. Since the operator norm depends on the norm of the underlying function space, so does the definition of the $\epsilon$-pseudospectrum. For resistive MHD, we use the energy norm, $\int (|v|^2 + |b|^2) \, dV$. In a coordinate system where $\|u\|^2$ is equal to the $L_2$ inner product, the operator norm corresponds to the largest singular value of the matrix representation of $A$. We denote the smallest singular value of $A - \lambda I$ by $\epsilon_{ud}(\lambda)$.

We call the test function in part (ii), $u$, an $\epsilon$-pseudomode and call $\lambda$ an $\epsilon$ – pseudoresonance. Part (iii) states that $\lambda \in \Lambda_\epsilon(A)$ is equivalent to the norm of the frequency response Green’s function at $\lambda$ being size $1/\epsilon$ or larger. Part (iv) says that the operator can be perturbed by a term of size $\epsilon$ such that the modified operator has an exact resonance at $\lambda$.

The equivalence of these four conditions is proven in Ref. 12, 13 and 16. In several excellent articles, the $\epsilon$-pseudospectrum is analyzed for the Orr-Sommerfeld equation\textsuperscript{12–15} and the convection diffusion equation\textsuperscript{17}.

In resistive MHD, the simple eigenvalue problem, $Au = \lambda u$, is replaced by the generalized eigenvalue problem, $Au = \lambda Mu$, where the weight matrix, $M$, is the matrix defined in the energy norm of the perturbation. In Appendix A, we give generalizations of Definition 1 to the generalized eigenvalue problem. When $M$ is self-adjoint, positive definite and bounded above and below, we can transform the problem into a standard eigenvalue problem: $A_F u' = \lambda u'$ where $F^*F = M$, $u' = Fu$, and $A_F \equiv F^{-1}A F^{-1}$, where $*$ denotes the adjoint operator. We then compute the $\epsilon$-pseudospectrum of the standard linear problem. This transformation gives the $\epsilon$-pseudospectrum for the generalized eigenvalue problem in the physically correct energy norm; however, $F^{-1}A F^{-1}$ is no longer a banded matrix. As a result, the computation of the singular value decomposition is very costly. Therefore, we consider a different generalization of the $\epsilon$-pseudospectrum, which replaces part (i) of Definition 1 with the smallest singular value of $A - \lambda M$. To correctly normalize this definition of the generalized $\epsilon$-pseudospectrum, we divide the singular values of $A - \lambda M$ by $\|M\|$. (See Appendix A.) We compute the boundary of the generalized
As a first step in resolving the Alfvén paradox, we show that for sufficiently small $\eta$, the continuous spectrum of ideal MHD is contained in the $\epsilon$ pseudospectrum. We begin by stating a lemma on singular sequences:

**Lemma 1:** The spectrum of a self-adjoint operator, $L$, consists of those complex numbers, $\lambda$, for which there exists a sequence of functions $U_n$ such that $||U_n|| = 1$ and $||L U_n - \lambda U_n|| \to 0$.

In Ref. 5, Laurence shows that the ideal MHD operator plus a multiple of the identity has a self-adjoint extension. (The multiple of the identity, $\sigma I$, is necessary to ensure positivity of the operator.) The ideal MHD operator is self-adjoint when the domain of the operator is restricted to perturbation of the form:

$$v(x, t = 0) = \xi, \quad b(x, t = 0) = \nabla \times (\xi \times B),$$

and the mprm is $\int |v|^2 + |b|^2 dV$.

Since the ideal MHD operator has a self-adjoint formulation, every value of $\lambda$ in the spectrum of the ideal MHD operator has a singular sequence of functions. In fluid dynamics, the ideal operator is not self-adjoint, and therefore singular sequences of functions need not exist for the spectrum of the inviscid Orr-Sommerfeld equation. In Ref. 4, Hameiri uses singular sequences to show that ballooning modes are part of the ideal MHD essential spectrum.

The domain of the ideal MHD operator differs from that of the resistive MHD operator, $L_\eta$, because the resistive operator involves more derivatives and requires more boundary conditions while the ideal MHD operator imposes the constraint that $b(x, t = 0) = \nabla \times [v(x, t = 0) \times B]$. We denote the ideal MHD operator, restricted to the intersection of the domains of the ideal and resistive MHD operators by $L_I$. This restriction amounts to considering the ideal MHD operator on the function space where $||\nabla \times \nabla \times b||^2$ is finite. The ideal MHD restriction: $b(x, t = 0) = \nabla \times [v(x, t = 0) \times B]$ remains in effect. For resistive MHD, we use the norm, $\int (|v|^2 + |b|^2) dV$, while ideal MHD is self-adjoint in a different norm, $\int |v|^2 dV$.

**Definition:** $\lambda$ is in the dissipative spectrum of the ideal MHD operator if and only if there is a singular sequence of functions in the intersection of the domains of the RMHD and ideal operators such that $||U_n|| = 1$ and $||L U_n - \lambda U_n|| \to 0$.

By Lemma 1, the dissipative spectrum of the ideal MHD operator is a subset of the spectrum of the ideal MHD operator. In ideal MHD, the singular function sequences are usually smooth functions which are localized near the resonance surface. In fact, we are unaware of any spectrum in ideal MHD where the singular sequence of test functions cannot be constructed in the function space of the resistive MHD operator. We introduce the terminology of “dissipative spectrum” in order to prove the following theorem:

**Theorem 1:** Let $\lambda$ be in the dissipative spectrum of the ideal MHD operator, $L_I$, for a bounded toroidal MHD equilibrium, and let $\epsilon > 0$ be given. Then there exists a critical value of resistivity, $\eta_{cr}$, such that $\lambda$ is contained in the $\epsilon$-pseudospectrum of resistive MHD for all $0 < \eta \leq \eta_{cr}$.
Proof: By Lemma 1, there is a sequence of test functions, \( U_n \), with \( ||U_n|| = 1 \), such that \( ||L_I U_n - \lambda U_n|| \to 0 \). For simplicity, we denote the resistive MHD operator by \( L_\eta U_n = L_I U_n + \eta \nabla \times \nabla \times b_n \) where \( b_n \) is the magnetic field component of \( U_n \). We apply criterion (ii) from the definition of \( \epsilon \)-spectrum.

\[ ||L_\eta U_n - \lambda U_n|| = ||L_I U_n + \eta \nabla \times \nabla \times b_n - \lambda U_n|| . \]

Using the Minkowski inequality, it follows

\[ ||L_\eta U_n - \lambda U_n|| \leq ||L_I U_n - \lambda U_n|| + \eta ||\nabla \times \nabla \times b_n|| . \]

We select \( U_n \) such that \( ||L_I U_n - \lambda U_n|| < \frac{\epsilon}{2} \) and select \( \eta_{cr} \) such that \( \eta_{cr} ||\nabla \times \nabla \times b_n|| < \frac{\epsilon}{2} \).

The spectrum of an arbitrary linear operator can be divided into three parts: the point spectrum, the continuous spectrum and the residual spectrum. Singular function sequences exist for the point spectrum and the continuous spectrum, but need not exist for the residual spectrum. Self-adjoint operators have no residual spectrum. To generalize Theorem 1 to the inviscid Orr-Sommerfeld equation, we need to require that \( L - \lambda I \) have a singular sequence in the domain of the viscid Orr-Sommerfeld equation.

Theorem 1 is a very general result, but it is a weak result in the sense that \( \eta \) scales as \( \epsilon \). In the next section, we derive a much stronger scaling, \( \epsilon \sim \exp(-1/\eta^{1/2}) \), for specific one-dimensional geometries.

V. WKBJ ANALYSIS

We restrict ourselves to the slab geometry with coordinates, \( x \equiv (x, y, z)^T \), and an equilibrium magnetic field, \( B(x) = B_y(x)\hat{y} + B_z(x)\hat{z} \). We consider perturbations of the form:

\[ b(x, t) = e^{\lambda t} e^{i(kz + my)}b(x) . \]

The equations can be decomposed into two separate eigenvalue problems, the transverse eigenmode equations and the longitudinal eigenmode problem. Our analysis will focus on the transverse equations which describe Alfvén waves. Eliminating the total pressure term in Eq. (1a) yields the transverse eigenmode equations\(^7\)\(^-\)\(^11\):

\[ H(x) \nabla^2 \psi - H''(x) \psi = \lambda \nabla^2 \xi , \quad \text{(2a)} \]

\[ \eta \nabla^2 \psi + H(x) \xi = \lambda \psi , \quad \text{(2b)} \]

where \( \psi \equiv b_x, \xi \equiv iv_x, \) and \( H(x) \equiv (k \cdot B(x))/\rho^{1/2} \). We impose perfectly conducting boundary conditions at \( x = x_a \) and \( x = x_b \): \( \xi(x_a) = \psi(x_a) = \xi(x_b) = \psi(x_b) = 0 \). The remainder of the article will concentrate on Eq. (2) for a single Fourier mode, \( e^{i(kz + my)} \). The \( \epsilon \)-pseudospectrum of the MHD operator is the union of the \( \epsilon \)-pseudospectra of all of the Fourier modes.
Equation (2) is a fourth order system of equations with two formal solutions that are asymptotic to the ideal MHD solutions. The remaining two formal solutions are constructed with the WKBJ expansion \[ \xi \sim (H(x)^2 - \lambda^2)^{-\frac{1}{4}} e^{\pm i\phi(x)} \] where

\[ \phi'(x) = \sqrt{\frac{H(x)^2 - \lambda^2}{i\lambda\eta}}. \]

The WKBJ phase function depends on the complex frequency, \( \omega \equiv -i\lambda \), and we will sometimes write \( \phi(x; \omega) \) to highlight this dependence. The WKBJ solutions oscillate and grow exponentially with a scale-length of \( \eta^{\frac{1}{2}} \).

Let \( \text{Im}[\phi(x)] \) have its minimum in the interior of the domain, at \( x = x_{mn} \) with \( x_a < x_{mn} < x_b \), and assume that \( \text{Im}[\phi(x_b)] \leq \text{Im}[\phi(x_a)] \). (Otherwise replace \( x_b \) with \( x_a \) in this paragraph.) We construct a formal \( \epsilon \)-pseudomode using the WKBJ formal solution, \( e^{i\phi(x)} \). To satisfy the boundary conditions, we add a low order polynomial (linear term) to \( e^{i\phi(x)} \). The size of this polynomial correction is \( O(e^{i[\phi(x_{mn})-\phi(x_b)]}) \). This formal \( \epsilon \)-pseudomode satisfies the RMHD operator up to a perturbation of size \( \epsilon \approx O(e^{i[\phi(x_{mn})-\phi(x_b)]}) \). Since \( \text{Im}[\phi(x_{mn}) - \phi(x_b)] \) is \( O(\frac{1}{\sqrt{\eta^2}}) \), the critical value of \( \epsilon \) scales as \( e^{-\frac{1}{\sqrt{\eta^2}}} \).

A similar construction is possible when \( \text{Im}[\phi(x)] \) has its maximum in the interior of the domain, using the WKBJ formal solution \( e^{-i\phi(x)} \). Summarizing our results, we have

Theorem 2: Consider a slab (or cylindrical) MHD equilibrium. For sufficiently small resistivity, there is a formal WKBJ \( \epsilon \)-pseudoeigenmode of the incompressible RMHD operator with \( \epsilon_{bd}(\lambda) \sim \exp(-1/\eta^{\frac{1}{2}}) \), provided that \( \text{Im}[\phi(x)] \) has a strict minimum (maximum) in the interior of the domain, i.e. \( e^{i\phi(x)} \) as a maximum (minimum). For monotone \( k \cdot B(x) \neq 0 \), the formal \( \epsilon \)-pseudoeigenmode exists in the half \( \lambda \)-annulus, \( \rho|\lambda|^2 = H(x)^2, \text{Re}[\lambda] < 0 \).

To prove the last sentence of Theorem 2, we use Property 1 of Ref. 9:

Property 1. For \( \text{Re}[\omega] \neq 0 \), \( \text{Im}[\phi'(x; \omega)] = 0 \) if and only if \( \rho|\omega|^2 = H(x)^2 \) and \( \text{Im}[\omega] \geq 0 \).

For monotone \( H(x) \) with \( H(x) \neq 0 \), Property 1 implies that \( \text{Im}[\phi](x; \omega) \) has either a maximum or a minimum in the interior of \( [x_a, x_b] \) when \( \omega \) is in the half annulus specified in Property 1.

In Theorem 2, we have discussed only formal solutions. A formal solution of Eq. (2) can fail to be asymptotic to an actual solution of Eq. (2) globally. Reference 9 provides a comprehensive discussion of the global validity of the WKBJ expansion. The formal solutions fail because the actual solution can pick up an exponentially growing solution, while the formal solution continues to decrease.
To construct the $\epsilon$-pseudomode in Theorem 2, we need to show that there is an actual solution which is exponentially larger in the interior than at the boundary. We restrict ourselves to complex analytic $H(x)$ profiles with a single transition point (where $\phi'(x) = 0$) in the complex domain around the real interval, $[x_a, x_b]$. From Property 1, this restriction corresponds to monotonically increasing $k \cdot B(x)$ profiles with $k \cdot B(x) \neq 0$ in the domain.

The validity of the formal WKBJ solutions depends on the geometry of level lines of $\text{Im}[\phi(x)]$. The anti-Stokes lines are the three curves of constant $\text{Im}[\phi(x)]$ which emerge from the transition point. Figure 1 displays the geometry of the anti-Stokes lines for different regions in the complex $\lambda$-plane.

From Ref. 9, we know that when one or no anti-Stokes line crosses $[x_a, x_b]$, the WKBJ expansion is valid. We consider the case where two different anti-Stokes lines cross $[x_a, x_b]$ at $x_1$ and $x_2$. Without loss of generality, we assume that $\text{Im}[\phi(x)]$ has its minimum at $x_{mn}$ with $x_a < x_1 < x_{mn} < x_2 < x_b$. From Ref. 9, the WKBJ expansion $e^{i\phi(x)}$ is valid in $[x_a, x_2]$, but we cannot exclude the possibility that the actual solution is not asymptotic to $e^{i\phi(x)} + ce^{2i\phi(x_2) - 2i\phi(x)}$ in the interval $[x_2, x_b]$, i.e. the actual solution grows exponentially in $[x_2, x_b]$. To construct an $\epsilon$-pseudomode, we need to require that $|e^{i\phi(x_{mn})}| \gg |e^{2i\phi(x_2) - 2i\phi(x)}|$, or in other words $\text{Im}[\phi(x_{mn}) + \phi(x_b) - 2\phi(x_2)] < 0$. Alternatively, we can use the WKBJ solution in $[x_1, x_b]$ and analytically continue it in $[x_a, x_1]$. In this case, the $\epsilon$-pseudomode construction is successful if $\text{Im}[\phi(x_{mn}) + \phi(x_a) - 2\phi(x_1)] < 0$.

In summary, our results are:

**Theorem 3.** Let $H(x)$ be a complex analytic, monotonic profile with $H(x) \neq 0$ in $[x_a, x_b]$ and a single transition point in the complex region around $[x_a, x_b]$. There is an actual $\epsilon$-pseudomode of the incompressible RMHD operator with $\epsilon \sim \exp(-\eta^{1/2})$ for complex eigenfrequencies which satisfy $\text{Im}[\omega] < 0$, $\rho|\omega|^2 = H(x)$, for a value in $[x_a, x_b]$, and one of the two conditions:

$$\text{Im}[\phi(x_{mn}) + \phi(x_b) - 2\phi(x_2)] < 0 ,$$

$$\text{Im}[\phi(x_{mn}) + \phi(x_a) - 2\phi(x_1)] < 0 ,$$

where $\text{Im}[\phi]$ has a minimum at $x_{mn}$, and the anti-Stokes lines are located at $x_1$ and $x_2$.

Our numerical results indicate that $\epsilon$-pseudomodes exist with $\epsilon \leq \exp(-\frac{1}{\eta^{1/2}})$ even when the conditions of Eq. (3) are not fulfilled. This frequency region is near the ideal MHD continuous spectrum. Thus it may be possible to construct $\epsilon$-pseudomodes in this region using a combination of the WKBJ formal solutions and the ideal MHD formal solutions.

**VI. NUMERICAL RESULTS**

Figure 2 shows the $\epsilon - \text{pseudospectrum}$ contours for Alfvén waves, computed by applying the singular value decomposition to $A - \lambda M$. A linear slab equilibrium with $H(x) = x$, $x_a = 0.2$, $x_b = 0.4$, and $\rho = 1$ is used. The Alfvén frequency, $\omega_A$, varies linearly from 0.2 to 0.4. The finite-element discretization is described in Appendix C. The number of radial cubic finite-elements used in the discretization
of the equation is 41, which is sufficient to resolve the structure of the solutions for $\eta = 10^{-4}$. In the same figure, we superimpose the resistive MHD eigenvalues.

Table 1 displays the RMHD eigenvalues for the finite-element discretization and the WKBJ approximation. For the WKBJ approximation, we have used the dispersion relation: $\phi(x_a) - \phi(x_b) = n\pi$. The good agreement between the numerical and the WKBJ eigenvalues demonstrates the accuracy of our numerical method.

Figure 3 compares the numerical and analytical contours of the $\epsilon$-pseudospectrum. The solid lines are the computed values of $\epsilon_{bd}$ and the dashed lines correspond to the WKBJ approximation presented in Appendix D (D9). The large $\epsilon$-pseudospectrum contours around the triple point indicate that these eigenvalues are very sensitive to perturbations. The similarity of the numerical and WKBJ contours show that the eigenvalue sensitivity is not due to the numerical discretization of the original equations, but rather is a property of the RMHD operator.

The discrepancy between the analytic and numeric $\epsilon$-pseudospectra occurs in the $\lambda$ region which is to the right of the triple point and which has two anti-Stokes lines intersect $[x_a, x_b]$. The calculation of the analytic $\epsilon$-pseudospectrum in Appendix D explicitly assumes that the WKBJ solutions are valid globally, and this assumption fails when two anti-Stokes lines intersect $[x_a, x_b]$. Thus, it is natural that some discrepancy occurs below the triple point in Fig. 3.

In Figure 4, we present a cross-section of the $\epsilon$-pseudospectrum at $Re[\lambda] = -0.1$ for different values of the resistivity. A comparison between the analytical and the numerical $\epsilon$-pseudospectrum is done in Figure 5, for $\eta = 10^{-4}$.

The most sensitive eigenvalues are those which are located near the triple point where the three branches of the different eigenvalues curves come together. The $\epsilon$-pseudospectrum contours expand rapidly around the triple point.

If the finite numerical accuracy is smaller than $\epsilon$ and the corresponding $\epsilon$-pseudospectrum contour extends into a large region, then the numerical code cannot properly resolve the eigenvalues inside the $\epsilon$-pseudospectrum contour, regardless of the number of grid points which are used in the discretization. For very small resistivity, the $\epsilon$-pseudospectrum is lower than machine roundoff in the half-annulus given in Theorem 2, and the resulting eigenvalues are scattered within this region.

**VII. INITIAL VALUE PROBLEM USING RMHD EIGENMODES**

For the Orr-Sommerfeld equation, a similar problem in fluid dynamics, it has been shown that the eigenmodes form a complete basis. Therefore, it is reasonable to believe that the resistive MHD eigenmodes are complete as well. Assuming that the resistive MHD eigenmodes form a complete basis, then an arbitrary initial perturbations will decay exponentially as $t \to \infty$. In contrast, perturbations in ideal MHD decay algebraically due to phase mixing.

The strong damping of the resistive eigenmodes has caused authors to question the completeness of the resistive spectrum and the significance of the resistive spectrum. Implicit in this argumentation is the belief that strong exponential damping of initial perturbations would occur on the Alfvénic time-scale. This intuition is based on normal operators and expansions of the solution in orthonormal eigenfunctions. We now show that the eigenfunctions of resistive MHD are nearly degenerate and that the condition number of the basis is very large, $\sim \exp(R_M^{1/2})$, where $R_M$
is the magnetic Reynolds number. The extended Bauer-Fike theorem gives a lower bound on the condition number of the eigenfunction basis in terms of the norm of the resolvent and the distance between the complex frequency, $\omega$, and the nearest eigenvalue. (See Appendix B of Ref. 12.) In Section IV, we have shown that the norm of the resolvent is $O(\exp(R_{M}^{1/2}))$ in the frequency half annulus. Most of this half annulus is a distance $O(1)$ from the eigenvalues. Thus the Bauer-Fike lower bound on the condition number is $O(\exp(R_{M}^{1/2}))$.

In expanding an initial perturbation in the RMHD eigenfunctions, the coefficients of the perturbation in the RMHD basis may be large, $O(\exp(R_{M}^{1/2}))$, due to the ill-conditioned RMHD basis. In the eigenfunction basis, each term individually damps on the Alfvén time-scale, but the coefficients are so large and the basis is so ill-conditioned that the combined sum behaves like the ideal solution does. Both analytical and numerical studies of RMHD have shown good agreement with ideal MHD on time scales which are long compared to the Alfvén time and short relative to the resistive time. Thus, we suggest that the resistive MHD eigenvectors are complete, but so poorly conditioned that they should not be used to interpret the temporal evolution on the ideal time-scale. From the bound on the condition number of the RMHD eigenmode basis, we believe that the eigenmode decomposition will only be useful for times of $O(R_{M}^{1/2})$.

VIII. TRANSIENT GROWTH OF INITIAL PERTURBATIONS

The other aspect of temporal evolution generated by non-normal operators is transient growth. Transient growth occurs when an initial perturbation grows in magnitude, as measured by the energy norm, before decaying. When the eigenfunctions are orthogonal and complete, and the system is stable, transient growth cannot occur. For non-normal operators, transient growth can occur due to the non-orthogonal nature of the eigenfunction basis. Butler and Farrell and Reddy, Schmid and Henningson have studied transient growth in the Orr-Sommerfeld equation and have found that initial perturbations can be amplified by factors of thousands. The transient growth of initial perturbations has been proposed as a mechanism by which fluctuations reach magnitudes which trigger nonlinear instabilities.

We show below that for a constant equilibrium current ($H(x) = x$) the energy of the perturbation is constant, so there is no transient growth in energy. In Refs. 12-16 and 19, optimization algorithms are given to determine the initial perturbation which experiences the largest transient growth. We have applied these algorithms to Eq. (2) with a variety of $H(x)$ profiles. For these profiles, we found only limited transient growth.

In ideal MHD, when the $\delta W$ energy principle is negative, there is an exponentially growing instability. When the $\delta W$ is positive, the total energy is constant:

$$\frac{d}{dt} \frac{1}{2} \int (|\partial_t \xi|^2 + \xi^\dagger F \xi) dV = 0$$
where $F$ is the $\delta W$ operator and $\xi$ is the displacement: $\partial_t \xi = \mathbf{v}$. The value of $\delta W$ is equal to the maximum possible amplification of the kinetic energy.

This bound on the kinetic energy growth is valid only for perturbations of the form $\mathbf{b}(x, t = 0) = \nabla \times (\mathbf{v}(x, t = 0) \times \mathbf{B})$. This restriction corresponds to considering perturbations which only displace the flux surface and do not change the topology of the magnetic field. In Appendix E, we reproduce a result of H. Grad’s which shows that linear in time growth occurs at the rational surface for more general perturbations with $\oint \mathbf{b} \cdot \nabla p^0 d\ell$ does not vanish on a rational flux surface. Thus we expect transient growth to be relevant for resonant perturbations with tearing mode parity. The Grad analysis addresses only growth in the ”supremum” norm and not with respect to the energy norm.

We now examine the energetics of transient growth:

\[
\frac{d}{dt} \frac{1}{2} \int (|\mathbf{v}|^2 + |\mathbf{b}|^2) dV = \int \mathbf{v} \cdot (\mathbf{J} \times \mathbf{b} + \mathbf{j} \times \mathbf{B} - \nabla p^1) + \mathbf{b} \cdot \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \mathbf{j})
\]

\[
= \int \mathbf{J} \cdot (\mathbf{v} \times \mathbf{b}) dV - \int \eta |\mathbf{j}|^2 dV - \oint [p^1 + \mathbf{B} \cdot \mathbf{b}] \mathbf{v} \cdot dS
\]

In deriving Eq. (6.1), we have used incompressibility. The first term can cause transient growth while the Ohmic heating term is purely stabilizing. The last term is the energy flux across the boundary and is zero by our boundary conditions. For reduced MHD in a slab geometry with constant current, $\mathbf{J}(x) = \mathbf{J}_0$, the energy transfer term, $\int \mathbf{J} \cdot (\mathbf{v} \times \mathbf{b}) dV$, reduces to the Poisson bracket of the corresponding flux functions, $\mathbf{J}_0 \int [\phi, \psi] dV$. In this case, the spatial integral of the energy transfer vanishes (as shown by integrating by parts.) Thus no transient growth occurs when $H(x)$ is linear.

In the Alfvén wave heating problem, the antenna at the boundary sends a net Poynting flux of energy into the plasma, and thereby forces the solution at the boundary. In Ref. 6, Kappraff and Tataronis show that the Alfvén wave heating problem has solutions which grow linearly in time until the Ohmic dissipation saturates the growth. Because the time integrated energy, which is transmitted by the antenna, grows linearly in time, it is not surprising that the kinetic energy grows initially as well. Thus, both the initial value problem and the Alfvén heating problem possess transiently growing solutions, but this growth is surprising for the initial value problem and is physically reasonable for forced problems such as Alfvén wave heating.

**IX. SUMMARY**

The resistive magnetohydrodynamics operator is nonnormal and its eigenvalues and eigenvectors are extremely sensitive to perturbation. Using the WKBJ approximation, we have shown that the entire stable half-annulus of complex frequencies with $\rho|\omega|^2 = |\mathbf{k} \cdot \mathbf{B}(x)|^2$ is in the $\epsilon$-pseudospectrum and that the critical value of $\epsilon$ for these frequencies scales as $\epsilon \sim \exp(-1/\eta^2)$. The frequency response Green’s
function is $O\left(\frac{1}{R}\right)$ in this half-annulus, and thus, the finite time response is influenced by all of the frequencies in the half-annulus.

We believe that the resistive magnetohydrodynamic eigenfunctions form a complete basis, but that this basis is so ill-conditioned, $O(\exp(\frac{1}{R^2M}))$, that it is not useful in describing the evolution of disturbances on the ideal magnetohydrodynamic timescale. From the scaling of the condition number of the resistive eigenmode basis, we believe that the eigenmode decomposition is only relevant for times of order $O(R^2M)$.

No transient growth occurs in a linear $k \cdot B(x)$ profile. When the current density is not constant, our preliminary computations indicate that weak transient amplification occurs. When rational surfaces are present and the initial perturbation has nonvanishing average of the normal magnetic field perturbation on the rational surface (a tearing mode-like structure), the ideal MHD perturbation grows linearly in time at the resonance surface. (See Appendix E.) However, the spatial extent of this perturbation may decay in time, and thus the total energy of the perturbation need not grow.

**Acknowledgement**

KSR acknowledges useful conversations with P. Deift, A. Lifshitz, L. N. Trefethen and H. Weitzner, and especially E. Hameiri and S. Reddy. We thank G. Spies for allowing us to reproduce Grad’s proof of algebraic growth from Spies’ lecture notes. The manuscript has benefited from critical readings by E. Hameiri, S. Reddy, and L. N. Trefethen. The authors acknowledge L. N. Trefethen for providing preprints. DNB acknowledges C. A. F. Varandas for the support given during the realization of this work. KSR’s work was performed under U.S. Department of Energy, Grant No. DE-FG02-86ER53223.

**APPENDIX A: GENERALIZED EPSILON PSEUDOSPECTRA**

We now state the various equivalent definitions of the generalized $\epsilon$-pseudospectra corresponding to the generalized eigenvalue problem, $A u = \lambda M u$. We refer the reader to Ref. 16 for proofs of the equivalences. We restrict our consideration to the finite dimensional case. We denote the spectrum of the generalized eigenvalue problem, $A e = \lambda Me$, by $\Lambda(A, M)$ and the resolvent set by $\rho(A, M)$.

Definition 2: Let $A$ and $M$ be closed linear operators with domain $D(A)$ and let $M$ be a positive definite self-adjoint operator such that there exists a constant $c > 0$ with $M \geq cI$. Let $\epsilon \geq 0$ be given. A complex number $z$ is in the $\epsilon$-pseudospectrum of $(A, M)$, which we denote by $\Lambda_\epsilon(A, M)$, if any of the following equivalent conditions is satisfied:

1. $\lambda$ is in the $\epsilon$-pseudospectrum of $F^{-*}AF^{-1}$, where $F^* = M$.
2. The smallest generalized $(M^{-1}, M)$ singular value of $A - \lambda M$ is less than or equal to $\epsilon$, i.e. $\epsilon \geq \min\{\mu(A - \lambda M, M^{-1}, M)\}$.
3. There exists $u \in D(A)$ such that $u^*Mu = 1$ and $u(A - \lambda M)^{-1}(A - \lambda M)u \leq \epsilon^2$.
4. $\lambda$ is in the generalized spectrum of $A + \epsilon F^*EF : (A + \epsilon F^*EF)u = \lambda Mu$, where $F^*F = M$ and $E$ satisfies $\|E\| \leq 1$. 

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(iii') there exists an operator, $H$, such that $\lambda$ is in the generalized spectrum of $A + \epsilon H : (A + \epsilon H)u = \lambda Mu$, where the matrix $H$ satisfies

$$
\max_{u \in \mathbb{C}^n} \frac{u^* H^* M^{-1} H u}{u^* M u} \leq 1.
$$

The equivalence of (i), (ii) and (iii) may be proved by simply transforming each of the properties from definition 1 to $F^{-*} A F^{-1}$.

In our numerical computations, we use a different generalization of $\epsilon$-pseudospectra. Our $M$-weighted $\epsilon$-pseudospectrum has the advantage that definitions (i)-(iii) are simpler than in Def. 2. However, the $M$-weighted $\epsilon$-pseudospectrum is not related to the standard $\epsilon$-pseudospectrum of Def. 1 through a change of variables. Thus the $M$-weighted $\epsilon$-pseudospectrum is not based on the MHD energy norm, and Definition 3 has no analog of (0) in Def. 2.

Definition 3: $(M$-weighted $\epsilon$-pseudospectrum). Let $A$ and $M$ be closed linear operators with domain $D(A)$ and let $M$ be a positive definite self-adjoint operator such that there exists a constant $c > 0$ with $M \geq cI$. Let $\epsilon \geq 0$ be given and define $\epsilon \equiv \epsilon ||M||$. A complex number $\lambda$ is in the $M$-weighted $\epsilon$-pseudospectrum of $A$, which we denote by $\Sigma_\epsilon(M | A)$, if one of the following equivalent conditions is satisfied:

(i) the smallest singular value of $A - \lambda M$ is less than or equal to $\epsilon$.

(ii) there exists $u \in D(A)$ such that $||u||^2 = 1$ and $||(A - \lambda M)u||^2 \leq \epsilon^2$,

(iii) $\lambda$ is a generalized eigenvalue of $A + \epsilon E$ w.r.t. $M$: $(A + \epsilon E)u = \lambda Mu$, where the matrix $E$ satisfies $||E|| \leq 1$.

The normalization, $\epsilon \equiv \epsilon ||M||$, allows Def. 3 to reduce to Def. 1 when $M$ is a multiple of the identity matrix.

APPENDIX B: PHASE INTEGRAL FOR THE LINEAR PROFILE

We evaluate the WKBJ phase function for the linear $H(x) = k \cdot B(x)$ profile. The phase integral becomes

$$
\phi(x; \lambda) = \int_{x_a}^{x_b} dx \sqrt{ax^2 + bx + c} = 
$$

$$
\left[ \frac{(2 \sqrt{ab} + 4a^2 x) \sqrt{\chi} - (b^2 - 4ac) \log\left( \frac{b}{2 \sqrt{a}} + \sqrt{ax + \sqrt{\chi}} \right)}{8a^2} \right]_{x_a}^{x_b}, \quad (B1)
$$

where

$$
\chi \equiv ax^2 + bx + c = -i \left( -\lambda^2 + \left( H(x_b) + (x - x_a) \frac{H(x_a) - H(x_b)}{x_b - x_a} \right)^2 \right). \quad (B2)
$$
The dispersion relation can be written in an implicit form $F(\lambda) = \phi(r_b, \lambda) - \phi(r_a, \lambda) - n\pi = 0$. Newton iteration is used to solve the dispersion relation. The derivative, $\frac{\partial F}{\partial \lambda}$, is computed analytically using Eq. (B1). The analytic eigenvalues are given in Table 1.

Equation (3) gives a sufficient criterion for the validity of the WKBJ expansion. For the linear $H(x)$ profile, Eq. (3) reduces to

$$
\left(\frac{b}{4a} + \frac{x_{mn} + x - 2x_2}{2}\right)\sqrt{\chi} - \frac{(b^2 - 4ac)}{8a^2} \log (\Pi) < 0, \quad (B3)
$$

with

$$
\Pi = \frac{(\frac{b}{2\sqrt{a}} + \sqrt{a}x_b + \sqrt{\chi})(\frac{b}{2\sqrt{a}} + \sqrt{a}x_{mn} + \sqrt{\chi})}{(\frac{b}{2\sqrt{a}} + \sqrt{a}x_2 + \sqrt{\chi})^2},
$$

and $x_{mn}$ satisfies $|H(x_{mn})| = |\lambda|$.

**APPENDIX C: FINITE-ELEMENT DISCRETIZATION**

In the Galerkin method, a weak form of Eq. (2) is constructed by multiplying the set of equations with an arbitrary test function and integrating over the domain of interest. In this case, we use a finite-element basis with the actual functions as test functions. We rewrite Eq. (2) in the reduced MHD form:

$$
\lambda \nabla_\perp^2 U_1 = B_0 \cdot \nabla (\nabla_\perp^2 A_1) + (\nabla A_1 \times \hat{z}) \cdot \nabla j_z,
$$

$$
\lambda A_1 = B_0 \cdot \nabla (U_1) + \eta \nabla_\perp^2 A_1, \quad (C1)
$$

where $A, U$ are the stream functions defined as, $B_1 = \nabla A_1 \times \hat{z} + B_\perp \hat{z}, \nu = \nabla U_1 \times \hat{z}$, and $\nabla_\perp = \nabla - \hat{z} \frac{\partial}{\partial z}$, the resulting generalized eigenvalue problem, $A \mathbf{u} = \lambda M \mathbf{u}$, has matrix elements:

$$
A(A_1, A_1) = -\eta \int \nabla_\perp A_1^* \cdot \nabla_\perp A_1 dV + \eta \int A_1^* \nabla_\perp A_1 \cdot \mathbf{n} dS,
$$

$$
A(A_1, U_1) = \int A_1^*(B_0 \cdot \nabla) U_1 dV.
$$

$$
A(U_1, A_1) = -\int U_1^*(B_1 \cdot \nabla) j_z dV + \int U_1^*(B_0 \cdot \nabla) \nabla_\perp^2 A_1 dV
$$

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\[= - \int U_1^* (B_1 \cdot \nabla) j z dV - \int \nabla \perp U_1^* \cdot (B_0 \cdot \nabla) \nabla \perp A_1 dV \]

\[- \int U_1^* \nabla \perp (B_0 \cdot \nabla) \nabla \perp A_1 dV + \int U_1^* (B_0 \cdot \nabla) \nabla \perp A_1 \cdot nds, \]

\[M(A_1, A_1) = \int A_1^* A_1 dV, \]

\[M(U_1, U_1) = - \int \nabla \perp U_1^* \cdot \nabla \perp U_1 dV + \int U_1^* \nabla \perp U \cdot \vec{n} dS. \quad (C2)\]

\(M\) is a Hermitian, positive-definite matrix. Due to the local support of the finite-elements, the integrand is nonzero only for neighboring points. A more detailed description of the numerical discretization is contained in Ref. 23.

For generalized eigenvalue problems, the QZ algorithm is usually recommended. However, we found that this algorithm is not stable for this kind of matrices. The error propagation is too large and the results are contaminated. The best results were obtained by inverting the \(M\) matrix and solving the eigenvalue problem \(M^{-1} A u = \lambda u\) applying the QR algorithm.

**APPENDIX D: ANALYTIC EPSILON-PSEUDOSPECTRUM**

To calculate the \(\epsilon\)-pseudospectrum analytically, we neglect the \(H''(x)\) term in Eq. (2a). By dropping this term, the WKBJ solutions decouple from the ideal MHD solutions and the RMHD operator, \((L - \lambda) \psi\) reduces to a second order equation:

\[L \lambda \psi = \eta \nabla^2 \psi - i \lambda \left[ 1 - \frac{H(x)^2}{\lambda^2} \right] \psi , \quad (D1)\]

where \(\lambda\) is now a nonlinear eigenvalue parameter. In general, this simplification is not valid for \(\lambda\) values which have two anti-Stokes line crossing the interval, \([x_a, x_b]\). (See Figure 1.) In this \(\lambda\)-region, the ideal solutions couple to the WKBJ solutions. For \(\lambda\) which have at most one anti-Stokes line crossing \([x_a, x_b]\), Ref. 9 shows that the formal solutions do not couple. In this case, our analysis of the \(\epsilon\)-pseudospectrum will be valid if the the \(\epsilon\)-pseudomode oscillates rapidly on the scale length of the WKBJ solutions.

To determine the \(\epsilon\)-pseudospectrum, we construct the Green’s function for Eq. (D1) using the WKBJ solutions:

\[\Psi_\pm (x) = (H(x)^2 - \lambda^2)^{-\frac{1}{2}} e^{\pm i \phi(x)} ... \quad (D2)\]
The Green’s function, $G(r, x)$, satisfies

$$\eta \nabla^2_r G(r, x) - i\lambda \left[ 1 - \frac{H(r)^2}{\lambda^2} \right] G(r, x) = \delta(r - x), \quad (D3)$$

with the boundary conditions: $G(x_a, x) = 0$ and $G(x_b, x) = 0$. We define the function $\chi(y, z) \equiv \Psi_+(y)\Psi_-(z) - \Psi_+(z)\Psi_-(y)$, and the functions, $\Psi_0(r, x)$ and $\Psi_1(r, x)$:

$$\Psi_0(r, x) = \frac{\chi(x_a, r)\chi(x_b, x)}{\chi(x_b, x_a)}, \quad (D4)$$

$$\Psi_1(r, x) = \frac{\chi(x_b, r)\chi(x_a, x)}{\chi(x_b, x_a)}. \quad (D5)$$

Note $\Psi_0(x_a, x) = 0$, $\Psi_1(x_b, x) = 0$ and $\Psi_0(x, x) = \Psi_1(x, x)$ at $r = x$. At $r = x$, the first derivatives satisfy the jump condition: $\partial_r \Psi_0(x, x) = \partial_r \Psi_1(x, x) + 1$. Furthermore,

$$\Psi_1(r, x) - \Psi_0(r, x) = \Psi_+(r)\Psi_-(x) - \Psi_-(r)\Psi_+(x) = \chi(r, x).$$

Thus, the Green’s function can be rewritten as,

$$G(r, x) = \Psi_0(r, x) + \chi(r, x) \Theta(r - x), \quad (D6)$$

where $\Theta$ is the Heaviside function. The resolvent, $\| (\lambda I - L)^{-1} \|$, is represented as

$$(\lambda I - L)^{-1} f = \int_{x_a}^{x_b} G(r, x) f(x) dx. \quad (D7)$$

To calculate the $\epsilon$-pseudospectrum, we determine the norm of the resolvent by maximizing $\| (\lambda I - L)^{-1} f \|/\| f \|$. When the WKBJ expansion is valid, $\chi(y, z) \sim O(\exp(i \int_y^z \phi'(x, \lambda) dx))$. Since $\chi(x_a, r)\chi(x_b, x) >> \chi(r, x)\chi(x_b, x_a)$, we neglect $\chi(r, x)\Theta(r - x)$ in Eq. (D6). Thus,

$$\| (\lambda I - L)^{-1} \| \simeq \sup_{f(x)} \left| \int_{x_a}^{x_b} dx \frac{\chi(x_a, r)\chi(x_b, x)f(x)}{\chi(x_b, x_a)\| f \|} \right|. \quad (D8)$$
\( \Psi_0(r, x) \) and \( \Psi_1(r, x) \) are largest at \( r = x_{mn} = x \). The supremum occurs for \( f(x) = \chi(x_b, x) \). The resulting expression for the \( \epsilon \)-pseudospectrum is

\[
\epsilon_{bd}(\lambda) \approx \frac{\chi(x_a, x_b)}{\chi(x_b, x_{max})\chi(x_a, x_{max})} .
\]

(D9)

This analysis is only valid when the formal WKBJ solutions are valid and \( \text{Im}[\phi'(x, \lambda)] \) has its minimum in the interval. (Property 1 shows that \( x_{mn} \) satisfies \( \rho|\lambda|^2 = H(x_{mn})^2 \).) Our analysis of the \( \epsilon \)-pseudospectrum is based on similar analysis for the convection diffusion problem given in Ref. 17.

**APPENDIX E: ALGEBRAIC GROWTH IN IDEAL MHD**

We now present a result of H. Grad’s which shows that the linearized circulation grows algebraically in time in ideal MHD for certain perturbations. Equations (E1)-(E2) are from Ref. 2. We consider a closed flux line and define the first order circulation as

\[
c(t) = \oint d\ell \cdot u_1 ,
\]

where the contour integral is along the field line. We now evaluate \( \frac{dc}{dt} \) and \( \frac{d^2c}{dt^2} \). Since the equilibrium is static, the path of integration does not move to lowest order, and the time derivative of \( c \) is obtained by just differentiating the integrand. Using the linearized equations of motion and the equilibrium equations and assuming that \( \nabla \rho_0 \times \nabla p_0 = 0 \), we compute

\[
\frac{dc}{dt} = \oint d\ell \cdot \frac{\partial u_1}{\partial t} = \frac{1}{\rho_0} \oint d\ell \cdot (-\nabla p_1 + j \times B + J \times b)
\]

\[
= \frac{1}{\rho_0} \oint d\ell \cdot (J \times b) = \frac{1}{\rho_0} \oint d\ell \frac{1}{|B|} B \cdot (J \times b)
\]

\[
= -\frac{1}{\rho_0} \oint d\ell \frac{1}{|B|} b \cdot (J \times B) = -\frac{1}{\rho_0} \oint d\ell \frac{1}{|B|} (b \cdot \nabla p_0)
\]

(E1)

and further

\[
\frac{d^2c}{dt^2} = -\frac{1}{\rho_0} \oint d\ell \frac{1}{|B|} \left( \frac{\partial b}{\partial t} \cdot \nabla p_0 \right) = -\frac{1}{\rho_0} \oint d\ell \frac{1}{|B|} \nabla p_0 \cdot \text{curl}(u \times B)
\]

\[
= -\frac{1}{\rho_0} \oint d\ell \frac{1}{|B|} \text{div}((u \times B) \times \nabla p_0) = -\frac{1}{\rho_0} \oint d\ell \frac{1}{|B|} \text{div}((u \cdot \nabla p_0)B)
\]

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\[
\rho_0 \oint d\ell \frac{1}{|B|} B \cdot \nabla (u_1 \cdot \nabla p_0) = -\frac{1}{\rho_0} \oint d\ell \cdot \nabla (u_1 \cdot \nabla p_0) = 0.
\]

(E2)

Hence \( dc/dt \) is constant. When \( \mathbf{b} \) has the form: \( \mathbf{b} = \text{curl}[\xi \times \mathbf{B}] \) with a single-valued vector field \( \xi \), then \( dc/dt = 0 \). (This is shown by using vector identities similar to those in Eq. (E2).) Since \( c(t) \) is growing linearly, the maximum of \( u(x,t) \) is growing at least linearly.

In slab geometry with a single helicity perturbation: \( \mathbf{b}(x,t) = e^{i(kz+my)} \mathbf{b}(x) \), the transient growth criterion of Eq. (E1) reduces to \( b_x(x_{\text{res}}) \neq 0 \), i.e. the perturbed normal flux at the resonance surface does not average to zero.

We strengthen Grad’s result by noting that the transiently growing ideal MHD solution is approximately a solution of the RMHD equations for small enough resistivity. Thus the RMHD equations will have transient growth in the supremum norm for tearing mode perturbations. Pointwise growth of the perturbation does not imply growth in the energy norm because the spatial extent of the perturbation can decrease. For the case of a linear profile, \( H(x) = x \), Sec. VIII shows that this profile is stable in the energy norm while growing in the supremum norm.

Landahl\(^{24}\) has shown that the inviscid Orr-Sommerfeld equation has solutions which grow linearly in time. Landahl’s unstable modes are global modes while the circulation instability of Eqs. (E1)-(E2) is localized on a field line.
References


**Table Caption** Comparison of WKBJ approximation and numerical calculation of the eigenvalues of resistive magnetohydrodynamics. FEM denotes the finite element calculation.

<table>
<thead>
<tr>
<th>n</th>
<th>FEM</th>
<th>WKBJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>$-0.06140 + 0.2951i$</td>
<td>$-0.06079 + 0.2948i$</td>
</tr>
<tr>
<td>13</td>
<td>$-0.07458 + 0.2915i$</td>
<td>$-0.07439 + 0.2915i$</td>
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<tr>
<td>14</td>
<td>$-0.08895 + 0.2872i$</td>
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<tr>
<td>15</td>
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<td>16</td>
<td>$-0.12024 + 0.2753i$</td>
<td>$-0.12006 + 0.2754i$</td>
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Figure Captions

Figure 1: Geometry of the anti-Stokes lines for different regions in the complex eigenvalue plane. Near the Alfvén continuum, two anti-Stokes lines cross the interval $[x_a, x_b]$, and the WKBJ expansion may not be valid.

Figure 2: Contours of the critical $\epsilon$ for resistive magnetohydrodynamics: A linear slab equilibrium is used: $H(x) = x$ with $x_a = 0.2$ and $x_b = 0.4$. Thus there is no resonance surface and the Alfvén frequency varies from 0.2 to 0.4. We use 41 cubic finite elements in the $x$ direction. For $\eta = 10^{-4}$, the radial structure of the eigenmodes is well resolved. The $\epsilon$ contours are computed by calculating the smallest singular value of $A - \lambda \mathbf{M}$ as a function of $\lambda$. The eigenvalues are marked with $x$.

Figure 3: Comparison of the WKBJ and numerical contours of the critical $\epsilon$ for $\eta = 10^{-4}$. The agreement is very good when a single anti-Stokes line crosses the interval $[x_a, x_b]$ and there is no discrete eigenvalue in the neighborhood of $\lambda$. When two anti-Stokes line cross the interval $[x_a, x_b]$, the WKBJ dispersion relation fails to capture the behavior of the critical $\epsilon$.

Figure 4: Dependence of the critical $\epsilon$ on the resistivity, $\eta$ for $\text{Re}[\lambda] = 0.1$ versus $\text{Im}[\lambda]$. The dependence of $\epsilon$ is clearly exponential in resistivity.

Figure 5: Comparison between the analytical and the numerical $\epsilon$–pseudospectrum for $\eta = 10^{-4}$. The agreement is very good. We plot the same $\lambda$ cross-section as in Fig. 4. From Fig. 3, we know that this agreement is much worse for $\text{Re}[\lambda] \leq 0.05$. 

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