CONVERGENCE OF HESTON TO SVI

JIM GATHERAL AND ANTOINE JACQUIER

Abstract. By an appropriate change of variables, we prove here that the SVI implied volatility parameterisation proposed in [2] and the large-time asymptotic of the Heston implied volatility derived in [1] do agree algebraically, thus confirming a conjecture proposed by J. Gatheral in [2] as well as proposing a simpler expression for the asymptotic implied volatility under the Heston model.

1. Notations

From [2], recall that the SVI parameterisation for the implied variance reads

$$\sigma_{SVI}^2(x) = \omega_1 \left(\omega_2 \rho x + \sqrt{(\omega_2 x + \rho)^2 + 1 - \rho^2}\right)/2, \quad \text{for all } x \in \mathbb{R}, \tag{1.1}$$

where x represents the log-moneyness, and consider the Heston model where the stock price process $(S_t)_{t\geq 0}$ follows the following dynamics:

$$\begin{aligned} \mathrm{d}S_t &= \sqrt{v_t} S_t \, \mathrm{d}W_t, \ S_0 \in \mathbb{R}_+^* \\ \mathrm{d}v_t &= \kappa(\theta - v_t)\,\mathrm{d}t + \sigma\sqrt{v_t}\,\mathrm{d}Z_t, \ v_0 \in \mathbb{R}_+^* \\ \mathrm{d}\langle W, Z\rangle_t &= \rho\,\mathrm{d}t, \end{aligned}$$

with $\rho \in [-1, 1]$, κ, θ, σ and v_0 are strictly positive real numbers satisfying $2\kappa\theta \geq \sigma^2$ (this is the Feller condition ensuring that the process $(v_t)_{t\geq 0}$ never reaches zero almost surely). We further make the following assumption as in [1], under which the Heston asymptotic implied volatility is derived.

Assumption 1.1. $\kappa - \rho\sigma > 0$.

Let us now consider the following change of variables in the SVI parameterisation, in terms of the Heston parameters,

$$\omega_1 := \frac{4\kappa\theta}{\sigma^2(1-\rho^2)}\left(\sqrt{(2\kappa - \rho\sigma)^2 + \sigma^2(1-\rho^2)} - (2\kappa - \rho\sigma)\right), \quad \text{and} \quad \omega_2 := \frac{\sigma}{\kappa\theta}. \tag{1.2}$$

Now, from [1], we know that the large-time asymptotics of the implied variance under the Heston model takes the following form

$$\sigma_\infty^2(x) = 2\left(2V^*(x) - x + 2\left(\mathbb{1}_{x\in(-\theta/2,\bar{\theta}/2)} - \mathbb{1}_{x\in\mathbb{R}\setminus(-\theta/2,\bar{\theta}/2)}\right)\sqrt{V^*(x)^2 - xV^*(x)}\right), \quad \text{for all } x \in \mathbb{R}, \tag{1.3}$$

where $\bar{\theta} := \kappa\theta/(\kappa - \rho\sigma)$, and the function $V^* : \mathbb{R} \to \mathbb{R}_+$ is defined by

$$V^*(x) := p^*(x)\,x - V(p^*(x)), \quad \text{for all } x \in \mathbb{R}, \tag{1.4}$$

where

$$\begin{aligned}
V(p) &:= \frac{\kappa\theta}{\sigma^2}\Big(\kappa - \rho\sigma p - d(p)\Big), \quad \text{for all } p \in (p_-, p_+),\\
d(p) &:= \sqrt{\left(\kappa - \rho\sigma p\right)^2 + \sigma^2 p\left(1 - p^2\right)}, \quad \text{for all } p \in (p_-, p_+),\\
p^*(x) &:= \frac{\sigma - 2\kappa\rho + (\kappa\theta\rho + x\sigma)\,\eta\left(x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2\right)^{-1/2}}{2\sigma\bar{\rho}^2}, \quad \text{for all } x \in \mathbb{R},\\
\eta &:= \sqrt{4\kappa^2 + \sigma^2 - 4\kappa\rho\sigma}, \quad \text{and} \quad p_\pm := \left(-2\kappa\rho + \sigma \pm \sqrt{\sigma^2 + 4\kappa^2 - 4\kappa\rho\sigma}\right) / \left(2\sigma\bar{\rho}^2\right).
\end{aligned}$$

2. Main result and proof

We now state and prove the main result of this note,

Proposition 2.1. *Under Assumption 1.1 and the change of variables* (1.2), *then* $\sigma^2_{SVI}(x) = \sigma^2_\infty(x)$ *for all* $x \in \mathbb{R}$.

Proof. Let us now introduce the following notations (we recall the definition of η for clarity):

$$\Delta(x) := \sqrt{\sigma^2 x^2 + 2\kappa\theta\rho\sigma x + \kappa^2\theta^2}, \quad \eta := \sqrt{4\kappa^2 + \sigma^2 - 4\kappa\rho\sigma}, \quad \text{and } \bar{\rho} := \sqrt{1-\rho^2}.$$

Under the change of variables (1.2), the SVI implied variance takes the form

$$\sigma^2_{SVI}(x) = \frac{2}{\sigma^2\bar{\rho}^2}\Big(\eta - (2\kappa - \rho\sigma)\Big)\Big(\kappa\theta + \rho\sigma x + \Delta(x)\Big), \quad \text{for all } x \in \mathbb{R}. \tag{2.1}$$

We now move on to simplify the expression for σ^2_∞ as written in (1.3). We first start by the expression for $V^*(x)$ appearing in (1.3). We have

$$V^*(x) = \frac{A(x)\,\Delta(x) + B(x)\eta}{2\sigma^2\bar{\rho}^2\Delta(x)},$$

with

$$A(x) := x\sigma^2 - 2x\kappa\rho\sigma - 2\kappa^2\theta + \kappa\theta\rho\sigma, \quad \text{and} \quad B(x) := 2x\sigma\kappa\theta\rho + x^2\sigma^2 + \kappa^2\theta^2\rho^2 + \kappa^2\theta^2\bar{\rho}^2.$$

Note that $B(x) = \Delta^2(x)$, so that $V^*(x) = (A(x) + \Delta(x)\eta) / \left(2\sigma^2\bar{\rho}^2\right)$. We further have

$$2V^*(x) - x = \frac{A(x) + \Delta(x)\eta - x\sigma^2\bar{\rho}^2}{\sigma^2\bar{\rho}^2} = \frac{\Delta(x)\eta - (2\kappa - \rho\sigma)(\kappa\theta + x\rho\sigma)}{\sigma^2\bar{\rho}^2}, \tag{2.2}$$

where we use the factorisation $A(x) - x\sigma^2\bar{\rho}^2 = -(2\kappa - \rho\sigma)(\kappa\theta + x\rho\sigma)$.

Now, back to (1.3), where we denote $\Phi(x) := V^*(x)^2 - xV^*(x)$. We have

$$\Phi(x) = \left(\frac{\Delta(x)\eta}{2\sigma^2\bar{\rho}^2}\right)^2 + \alpha(x)\,\Delta(x) + \beta(x),$$

where

$$\alpha(x) := -\frac{\eta\,(2\kappa - \rho\sigma)(\kappa\theta + x\rho\sigma)}{2\sigma^4\bar{\rho}^4}, \quad \text{and} \quad \beta(x) := \frac{1}{4\sigma^4\bar{\rho}^4}\left\{(2\kappa - \rho\sigma)^2(\kappa\theta + x\rho\sigma)^2 - x^2\sigma^4\bar{\rho}^4\right\}.$$

We now use the following factorisations:

$$\Delta^2(x) = (\kappa\theta + x\rho\sigma)^2 + x^2\sigma^2\bar{\rho}^2, \quad \text{and} \quad \eta^2 = (2\kappa - \rho\sigma)^2 + \sigma^2\bar{\rho}^2, \tag{2.3}$$

so that we can write $\beta(x) = \left(4\sigma^4\bar{\rho}^4\right)^{-1}\left((2\kappa-\rho\sigma)^2\Delta^2(x) - x^2\sigma^2\bar{\rho}^2\eta^2\right)$ and hence

$$\begin{aligned}
\Phi(x) &= \frac{1}{4\sigma^4\bar{\rho}^4}\left\{\left[(2\kappa-\rho\sigma)^2+\sigma^2\bar{\rho}^2\right]\Delta^2(x) + a(x)\Delta(x) + \left(\eta^2-\sigma^2\bar{\rho}^2\right)\left(\Delta^2(x) - x^2\sigma^2\bar{\rho}^2\right) - x^2\sigma^4\bar{\rho}^4\right\} \\
&= \frac{1}{4\sigma^4\bar{\rho}^4}\left\{(2\kappa-\rho\sigma)^2\Delta^2(x) + a(x)\Delta(x) + \eta^2(\kappa\theta+x\rho\sigma)^2\right\} \\
&= \frac{1}{4\sigma^4\bar{\rho}^4}\left\{\eta(\kappa\theta+x\rho\sigma) - (2\kappa-\rho\sigma)\Delta(x)\right\}^2 \qquad (2.4)
\end{aligned}$$

where, for convenience, we denote $a(x) := 4\sigma^4\bar{\rho}^4\alpha(x)$. To complete the proof, we need to take the square root of $\Phi(x)$, i.e. we need to study the sign of the expression under the square in (2.4). Using again (2.3), we have

$$\begin{aligned}
\eta(\kappa\theta+x\rho\sigma) - (2\kappa-\rho\sigma)\Delta(x) &= (\kappa\theta+x\rho\sigma)\sqrt{(2\kappa-\rho\sigma)^2+\sigma^2\bar{\rho}^2} - (2\kappa-\rho\sigma)\sqrt{(\kappa\theta+x\rho\sigma)^2+x^2\sigma^2\bar{\rho}^2} \\
&= \sqrt{\gamma(x)+\sigma^2\bar{\rho}^2(\kappa\theta+x\rho\sigma)^2} - \sqrt{\gamma(x)+x^2\sigma^2\bar{\rho}^2(2\kappa-\rho\sigma)^2},
\end{aligned}$$

where $\gamma(x) := (2\kappa-\rho\sigma)^2(\kappa\theta+x\rho\sigma)^2$. Now, because $\gamma(x) \geq 0$ for all $x \in \mathbb{R}$, then the sign of this whole expression is simply given by the sign of the difference $\psi(x) := \sigma^2\bar{\rho}^2(\kappa\theta+x\rho\sigma)^2 - x^2\sigma^2\bar{\rho}^2(2\kappa-\rho\sigma)^2$. Note further that we actually have $\psi(x) = \kappa\sigma^2\bar{\rho}^2(2x+\theta)(2x\rho\sigma+\kappa\theta-2\kappa x)$, that this polynomial has exactly two real roots $-\theta/2$ and $\bar{\theta}/2$, and that its second-order coefficient reads $-4\kappa\sigma^2\bar{\rho}^2(\kappa-\rho\sigma) < 0$ under Assumption 1.1. So, plugging (2.2) and (2.4) into (1.3), we exactly obtain (2.1) and the proposition follows. □

References

[1] M. Forde, A. Jacquier, A. Mijatovic (2009), Asymptotic formulae for implied volatility in the Heston model. Submitted.

[2] J. Gatheral (2004), A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives. Presentation at Global Derivatives & Risk Management, Madrid, May 2004, available at www.math.nyu.edu/fellows fin math/gatheral/madrid2004.pdf.

Bank of America Merrill Lynch and the Courant Institute, NYU

Department of Mathematics, Imperial College London and Zeliade Systems, Paris

E-mail address: ajacquie@imperial.ac.uk.