

# No-Dynamic-Arbitrage and Market Impact

Jim Gatheral



Cornell University Financial Engineering Seminar  
April 22, 2009

# Market impact and its estimation

- Our aim is to make a connection between the shape of the market impact function and the decay of market impact.
- Market impact is estimated in practice by aggregating all executions of a certain type, for example all VWAP executions.
- We will assume that price impacts are estimated as unconditional averages.
  - We average over different market conditions.
  - We average buys and sells (with appropriate sign adjustments).
- This accurately mimics the estimation of market impact functions in practice (cf Almgren 2005 for example).

# Model setup

- We suppose that the stock price  $S_t$  at time  $t$  is given by

$$S_t = S_0 + \int_0^t f(\dot{x}_s) G(t-s) ds + \int_0^t \sigma dZ_s \quad (1)$$

where  $\dot{x}_s$  is our rate of trading in dollars at time  $s < t$ ,  $f(\dot{x}_s)$  represents the impact of trading at time  $s$  and  $G(t-s)$  is a decay factor.

- $S_t$  follows an arithmetic random walk with a drift that depends on the accumulated impacts of previous trades.
- The cumulative impact of (others') trading is implicitly in  $S_0$  and the noise term.
- Drift is ignored.
  - Drift is a lower order effect.
  - We are averaging buys and sells.

## Model setup continued

- We refer to  $f(\cdot)$  as the *instantaneous market impact function* and to  $G(\cdot)$  as the decay kernel.
- (1) is a generalization of processes previously considered by Almgren, Bouchaud and Obizhaeva and Wang.
- (1) corresponds to the “bare propagator” formulation of Bouchaud et al. rather than the state-dependent formulation of Farmer, Lillo et al.

## Model as limit of discrete time process

- The continuous time process (1) can be viewed as a limit of a discrete time process (see Bouchaud et al. for example):

$$S_t = \sum_{i < t} f(\delta x_i) G(t - i) + \text{noise}$$

where  $\delta x_i = \dot{x}_i \delta t$  is the quantity traded in some small time interval  $\delta t$  characteristic of the stock, and by abuse of notation,  $f(\cdot)$  is the market impact function.

- $\delta x_i > 0$  represents a purchase and  $\delta x_i < 0$  represents a sale.
- $\delta t$  could be thought of as  $1/\nu$  where  $\nu$  is the trade frequency.
- Increasing the rate of trading  $\dot{x}_i$  is equivalent to increasing the quantity traded each  $\delta t$ .

# Price impact and slippage

- The cost of trading can be decomposed into two components:
  - The impact of our trading on the market price (the mid-price for example).
    - We refer to this effect as *price impact*.
  - Frictions such as effective bid-ask spread that affect only our execution price.
    - We refer to this effect as *slippage*. For small volume fractions, we can think of slippage as being proxied by VWAP slippage.
- In what follows, we will neglect slippage.
  - The inequality relationships we derive will all be weakened in practice to the extent that slippage becomes important.

## Cost of trading

- Denote the number of shares outstanding at time  $t$  by  $x_t$ . Then from (1), neglecting slippage, the cost  $C[\Pi]$  associated with a given trading strategy  $\Pi = \{x_t\}$  is given by

$$C[\Pi] = \int_0^T \dot{x}_t dt \int_0^t f(\dot{x}_s) G(t-s) ds \quad (2)$$

- The  $dx_t = \dot{x}_t dt$  shares liquidated at time  $t$  are traded on average at a price

$$S_t = S_0 + \int_0^t f(\dot{x}_s) G(t-s) ds$$

which reflects the residual cumulative impact of all prior trading.

# Almgren et al.

- In our notation, the temporary component of Almgren's model corresponds to setting  $G(t - s) = \delta(t - s)$  and  $f(v) = \eta \sigma v^\delta$  with  $\delta = 0.6$ .
- In this model, temporary market impact decays instantaneously. Our trading affects only the price of our own executions; other executions are not affected.
- The cost of trading becomes:

$$C[\Pi] = \int_0^T \dot{x}_t dt \int_0^t f(\dot{x}_s) G(t - s) ds = \eta \sigma \int_0^T \dot{x}_t^{1+\delta} dt$$

# Obizhaeva and Wang

- In the setup of Obizhaeva and Wang, we have  $G(t - s) = \exp\{-\rho(t - s)\}$  and  $f(v) \propto v$ .
- In this model, market impact decays exponentially and instantaneous market impact is linear in the rate of trading.
- The cost of trading becomes:

$$\begin{aligned} C[\Pi] &= \int_0^T \dot{x}_t dt \int_0^t f(\dot{x}_s) G(t - s) ds \\ &\propto \int_0^T \dot{x}_t dt \int_0^t \dot{x}_s \exp\{-\rho(t - s)\} ds \end{aligned}$$

# Bouchaud et al.

- In the setup of Bouchaud et al., we have  $f(v) \propto \log(v)$  and

$$G(t-s) \propto \frac{l_0}{(l_0 + t - s)^\gamma}$$

with  $\gamma \approx (1 - \alpha)/2$  where  $\alpha$  is the exponent of the power law of autocorrelation of trade signs.

- In this model, market impact decays as a power law and instantaneous market impact is concave in the rate of trading.
- The cost of trading becomes:

$$\begin{aligned} C[\Pi] &= \int_0^T \dot{x}_t dt \int_0^t f(\dot{x}_s) G(t-s) ds \\ &\propto \int_0^T \dot{x}_t dt \int_0^t \frac{\log(\dot{x}_s)}{(l_0 + t - s)^\gamma} ds \end{aligned}$$

# The principle of No Dynamic Arbitrage

A trading strategy  $\Pi = \{x_t\}$  is a *round-trip trade* if

$$\int_0^T \dot{x}_t dt = 0$$

We define a *price manipulation* to be a round-trip trade  $\Pi$  whose expected cost  $C[\Pi]$  is negative.

The principle of no-dynamic-arbitrage

Price manipulation is not possible.

Corollary

Pump and dump schemes cannot make money on average

# Pump and Dump Schemes

(From <http://www.sec.gov/answers/pumpdump.htm>)

## Definition

“Pump and dump” schemes, also known as “hype and dump manipulation”, involve the touting of a company’s stock (typically microcap companies) through false and misleading statements to the marketplace. After pumping the stock, fraudsters make huge profits by selling their cheap stock into the market.



## Permanent impact

Suppose we trade into a position at the rate  $+v$  and out at the same  $-v$ . If market impact is permanent, without loss of generality,  $G(\cdot) = 1$  and the cost of trading becomes

$$\begin{aligned} C[\Pi] &= v f(v) \left\{ \int_0^{T/2} dt \int_0^t ds - \int_{T/2}^T dt \int_0^{T/2} ds \right\} \\ &\quad + v f(-v) \int_{T/2}^T dt \int_{T/2}^t ds \\ &= v \frac{T^2}{8} \{-f(-v) - f(v)\} \end{aligned}$$

- If  $f(v) \neq -f(-v)$ , price manipulation is possible.
- No-dynamic-arbitrage thus imposes that if market impact is permanent,  $f(v) = -f(-v)$ .
- We henceforth assume that  $f(v) = -f(-v)$ .

## A specific strategy

Consider a strategy where shares are accumulated at the (positive) constant rate  $v_1$  and then liquidated again at the (positive) constant rate  $v_2$ . According to equation (2), the cost of this strategy is given by  $C_{11} + C_{22} - C_{12}$  with

$$\begin{aligned} C_{11} &= v_1 f(v_1) \int_0^{\theta T} dt \int_0^t G(t-s) ds \\ C_{22} &= v_2 f(v_2) \int_{\theta T}^T dt \int_{\theta T}^t G(t-s) ds \\ C_{12} &= v_2 f(v_1) \int_{\theta T}^T dt \int_0^{\theta T} G(t-s) ds \end{aligned} \quad (3)$$

where  $\theta$  is such that  $v_1 \theta T - v_2 (T - \theta T) = 0$  so

$$\theta = \frac{v_2}{v_1 + v_2}$$

## Special case: Trade in and out at the same rate

One might ask what happens if we trade into, then out of a position at the same rate  $v$ . If  $G(\cdot)$  is strictly decreasing,

$$\begin{aligned} C[\Pi] &= v f(v) \left\{ \int_0^{T/2} dt \int_0^t G(t-s) ds + \int_{T/2}^T dt \int_{T/2}^t G(t-s) ds \right. \\ &\quad \left. - \int_{T/2}^T dt \int_0^{T/2} G(t-s) ds \right\} \\ &= v f(v) \left\{ \int_0^{T/2} dt \int_0^t [G(t-s) - G(t+T/2-s)] ds \right. \\ &\quad \left. + \int_{T/2}^T dt \int_{T/2}^t [G(t-s) - G(T-s)] ds \right\} > 0 \end{aligned}$$

- We conclude that if there is arbitrage, it must involve trading in and out at different rates.

## Exponential decay

Suppose that the decay kernel has the form

$$G(\tau) = e^{-\rho\tau}$$

Then, explicit computation of all the integrals in (3) gives

$$\begin{aligned} C_{11} &= v_1 f(v_1) \frac{1}{\rho^2} \left\{ e^{-\rho\theta T} - 1 + \rho\theta T \right\} \\ C_{12} &= v_2 f(v_1) \frac{1}{\rho^2} \left\{ 1 + e^{-\rho T} - e^{-\rho\theta T} - e^{-\rho(1-\theta)T} \right\} \\ C_{22} &= v_2 f(v_2) \frac{1}{\rho^2} \left\{ e^{-\rho(1-\theta)T} - 1 + \rho(1-\theta)T \right\} \end{aligned} \quad (4)$$

We see in particular that the no-arbitrage principle forces a relationship between the instantaneous impact function  $f(\cdot)$  and the decay kernel  $G(\cdot)$ .

## Exponential decay

After making the substitution  $\theta = v_2/(v_1 + v_2)$  and imposing the principle of no-dynamic-arbitrage, we obtain

$$\begin{aligned} & v_1 f(v_1) \left[ e^{-\frac{v_2 \rho}{v_1+v_2}} - 1 + \frac{v_2 \rho}{v_1 + v_2} \right] \\ & + v_2 f(v_2) \left[ e^{-\frac{v_1 \rho}{v_1+v_2}} - 1 + \frac{v_1 \rho}{v_1 + v_2} \right] \\ & - v_2 f(v_1) \left[ 1 + e^{-\rho} - e^{-\frac{v_1 \rho}{v_1+v_2}} - e^{-\frac{v_2 \rho}{v_1+v_2}} \right] \geq 0 \end{aligned} \quad (5)$$

where, without loss of generality, we have set  $T = 1$ . We note that the first two terms are always positive so price manipulation is only possible if the third term ( $C_{12}$ ) dominates the others.

Example:  $f(v) = \sqrt{v}$

Let  $v_1 = 0.2$ ,  $v_2 = 1$ ,  $\rho = 1$ . Then the cost of liquidation is given by

$$C = C_{11} + C_{22} - C_{12} = -0.001705 < 0$$

Since  $\rho$  really represents the product  $\rho T$ , we see that for any choice of  $\rho$ , we can find a combination  $\{v_1, v_2, T\}$  such that a round trip with no net purchase or sale of stock is profitable. We conclude that if market impact decays exponentially, no arbitrage excludes a square root instantaneous impact function.

Can we generalize this?

## Expansion in $\rho$

Expanding expression (5) in powers of  $\rho$ , we obtain

$$\frac{v_1 v_2 [v_1 f(v_2) - v_2 f(v_1)] \rho^2}{2(v_1 + v_2)^2} + O(\rho^3) \geq 0$$

We see that arbitrage is always possible for small  $\rho$  unless  $f(v)$  is linear in  $v$ .

Taking the limit  $\rho \rightarrow 0^+$ , we obtain

### Corollary

Non-linear permanent market impact is inconsistent with the principle of no-dynamic-arbitrage.

# Exponential decay of market impact and arbitrage

## Lemma

*If temporary market impact decays exponentially, price manipulation is possible unless  $f(v) \propto v$ .*

- Empirically, market impact is concave in  $v$  for small  $v$ .
- Also, market impact must be convex for very large  $v$ 
  - Imagine submitting a sell order for 1 million shares when there are bids for only 100,000.
- We conclude that the principle of no-dynamic-arbitrage excludes exponential decay of market impact for any reasonable instantaneous market impact function  $f(\cdot)$ .

## Nonlinear market impact example

### 'Fat finger' trade costs Tokyo shares boss his job - 20 December, 2005

The head of the Tokyo Stock Exchange has resigned over a trading error that has damaged confidence in Asia's largest stock market.

President Takuo Tsurushima and two other senior exchange executives apologised to investors yesterday for the "confusion" caused earlier this month when the TSE's trading system failed to block a sell order for 610,000 shares at 1 Yen apiece.

The fiasco was caused by a so-called "fat-finger" keyboard error by an employee of Mizuho Securities, who had only intended to sell one share in a manpower recruitment firm, J-Com, for 610,000 Yen. The sell order was more than 40 times larger than the firm's 14,500 outstanding shares.

## Linear permanent market impact

If  $f(v) = \eta v$  for some  $\eta > 0$  and  $G(t - s) = 1$ , the cost of trading becomes

$$C[\Pi] = \eta \int_0^T \dot{x}_t dt \int_0^t \dot{x}_s ds = \frac{\eta}{2} (x_T - x_0)^2$$

The trading cost per share is then given by

$$\frac{C[\Pi]}{|x_T - x_0|} = \frac{\eta}{2} |x_T - x_0|$$

which is *independent* of the details of the trading strategy (depending only on the initial and final positions) and linear in the net trade quantity.

## Power-law decay

Suppose now that the decay kernel has the form

$$G(t-s) = \frac{1}{(t-s)^\gamma}, \quad 0 < \gamma < 1$$

Then, explicit computation of all the integrals in (3) gives

$$\begin{aligned} C_{11} &= v_1 f(v_1) \frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)} \theta^{2-\gamma} \\ C_{22} &= v_2 f(v_2) \frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)} (1-\theta)^{2-\gamma} \\ C_{12} &= v_2 f(v_1) \frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)} \{1 - \theta^{2-\gamma} - (1-\theta)^{2-\gamma}\} \end{aligned} \tag{6}$$

## Power-law decay

According to the principle of no-dynamic-arbitrage, substituting  $\theta = v_2/(v_1 + v_2)$ , we must have

$$f(v_1) \{v_1 v_2^{1-\gamma} - (v_1 + v_2)^{2-\gamma} + v_1^{2-\gamma} + v_2^{2-\gamma}\} + f(v_2) v_1^{2-\gamma} \geq 0 \quad (7)$$

- If  $\gamma = 0$ , the no-arbitrage condition (7) reduces to

$$f(v_2) v_1 - f(v_1) v_2 \geq 0$$

so again, permanent impact must be linear.

- If  $\gamma = 1$ , equation (7) reduces to

$$f(v_1) + f(v_2) \geq 0$$

So long as  $f(\cdot) \geq 0$ , there is no constraint on  $f(\cdot)$  when  $\gamma = 1$ .

## The limit $v_1 \ll v_2$ and $0 < \gamma < 1$

In this limit, we accumulate stock much more slowly than we liquidate it. Let  $v_1 = \epsilon v$  and  $v_2 = v$  with  $\epsilon \ll 1$ . Then, in the limit  $\epsilon \rightarrow 0$ , with  $0 < \gamma < 1$ , equation (7) becomes

$$\begin{aligned} & f(\epsilon v) \{ \epsilon - (1 + \epsilon)^{2-\gamma} + \epsilon^{2-\gamma} + 1 \} + f(v) \epsilon^{2-\gamma} \\ \sim & -f(\epsilon v)(1 - \gamma)\epsilon + f(v)\epsilon^{2-\gamma} \geq 0 \end{aligned}$$

so for  $\epsilon$  sufficiently small we have

$$\frac{f(\epsilon v)}{f(v)} \leq \frac{\epsilon^{1-\gamma}}{1 - \gamma} \quad (8)$$

If the condition (8) is not satisfied, price manipulation is possible by accumulating stock slowly, maximally splitting the trade, then liquidating it rapidly.

## Power-law impact: $f(v) \propto v^\delta$

If  $f(v) \sim v^\delta$  (as per Almgren et al.), the no-dynamic-arbitrage condition (8) reduces to

$$\epsilon^{1-\gamma-\delta} \geq 1 - \gamma$$

and we obtain

Small  $v$  no-dynamic-arbitrage condition

$$\gamma + \delta \geq 1$$

## Log impact: $f(v) \propto \log(v/v_0)$

- $v_0$  should be understood as a minimum trading rate.
  - One could think of one share every trade as being the minimum rate.
  - For example, for a stock that trades 10 million shares a day, 10,000 times, the average trade size is 1,000. That implies  $v_0 = 0.10\%$ .

- Noting that

$$\log v = \lim_{\delta \rightarrow 0} \frac{v^\delta - 1}{\delta},$$

we would guess that there is arbitrage for all  $\gamma < 1$ .

- In practice, it depends on how small  $v_0$  is.
  - For example, substituting  $v_0 = 0.001$ ,  $v_1 = 0.15$ ,  $v_2 = 1.0$  and  $\gamma = 1/2$  into the arbitrage condition (7) with  $f(v) = \log(v/v_0)$  gives a negative cost (*i.e.* manipulation).
  - Formally, for every  $\gamma < 1$ , we can find  $v_0$  small enough to allow price manipulation.

## Cost of VWAP with power-law market impact and decay

From equation (6), the cost of an interval VWAP execution with duration  $T$  is proportional to

$$C = v f(v) T^{2-\gamma}$$

Noting that  $v = n/T$ , and putting  $f(v) \propto (v/V)^\delta$ , the cost per share is proportional to

$$\left(\frac{n}{V}\right)^\delta T^{1-\gamma-\delta}$$

If  $\gamma + \delta = 1$ , the cost per share is *independent* of  $T$  and in particular, if  $\gamma = \delta = 1/2$ , the impact cost per share is proportional to  $\sqrt{n/V}$ , which is the well-known square-root formula for market impact as described by, for example, Grinold and Kahn.

# A heuristic derivation of the square-root market impact formula

- Suppose each trade impacts the mid-log-price of the stock by an amount proportional to  $\sqrt{n_i}$  where  $n_i$  is the size of the  $i$ th trade.
- Then the change in mid-price over one day is given by

$$\Delta P = \sum_i^N \eta \epsilon_i \sqrt{n_i}$$

where  $\eta$  is the coefficient of market impact,  $\epsilon_i$  is the sign of the  $i$ th trade and  $N$  is the (random) number of trades in a day.

- Note that both the number of trades and the size of each trade in a given time interval are random.

## Derivation continued

- If  $N$ ,  $\epsilon_i$  and  $n_i$  are all independent, the variance of the one-day price change is given by

$$\sigma^2 := \text{Var}(\Delta P) = \eta^2 \mathbb{E}[N] \mathbb{E}[n_i] = \eta^2 V$$

where  $V$  is the average daily volume.

- It follows that

$$|\Delta P_i| = \eta \sqrt{n_i} = \sigma \sqrt{\frac{n_i}{V}}$$

which is the familiar square-root market impact formula.

# Why $\sqrt{n}$ ?

An inventory risk argument:

- A market maker requires an excess return proportional to the risk of holding inventory.
- Risk is proportional to  $\sigma \sqrt{T}$  where  $T$  is the holding period.
- The holding period should be proportional to the size of the position.
- So the required excess return must be proportional to  $\sqrt{n}$ .

## The square-root formula, $\gamma$ and $\delta$

- The square-root market impact formula has been widely used in practice for many years.
- If correct, this formula implies that the cost of liquidating a stock is independent of the time taken.
  - Fixing market volume and volatility, impact depends only size.
- We can check this prediction empirically.
  - See for example Engle, Ferstenberg and Russell, 2008.
- Also, according to Almgren,  $\delta \approx 0.6$  and according to Bouchaud  $\gamma \approx 0.4$ .

### Empirical observation

$$\delta + \gamma \approx 1!$$

# Bouchaud's power-law decay argument

- As before, assume that over one day

$$\Delta P = \sum_i^N \eta \epsilon_i \sqrt{n_i}$$

- The previous heuristic proof of the square-root model assumed that  $\text{Cov}[\epsilon_i, \epsilon_j] = 0$  if  $i \neq j$  and that all market impact is permanent.
- Empirically, we find that autocorrelation of trade signs shows power-law decay with a small exponent  $\alpha$  (very slow decay).

# Bouchaud's power-law decay argument continued

$$\begin{aligned}\text{Var}[\Delta P] &= \eta^2 \text{Var} \left[ \sum_i^N \epsilon_i \sqrt{n_i} \right] \\ &= \eta^2 \left\{ N \text{Var}[\sqrt{n_i}] + \sum_{i \neq j} \text{Cov}[\epsilon_i, \epsilon_j] \right\} \\ &\approx \eta^2 \left\{ N \text{Var}[\sqrt{n_i}] + \frac{2 C_1}{(2 - \alpha)(1 - \alpha)} \mathbb{E}[\sqrt{n}]^2 N^{2 - \alpha} \right\} \\ &\sim N^{2 - \alpha} \text{ as } N \rightarrow \infty\end{aligned}$$

- Empirically, we find that, to a very good approximation,  $\text{Var}[\Delta P] \propto N$ .
  - Otherwise returns would be serially correlated.
- The only way to reconcile these observations is to have market impact decay as a power law.

## Computation of daily variance with power-law decay

- Assuming market impact decays as  $1/T^\gamma$ , we have

$$\begin{aligned} \text{Var}[\Delta P] &= \eta^2 \text{Var} \left[ \sum_i^N \frac{\epsilon_i \sqrt{n_i}}{(N-i)^\gamma} \right] \\ &= \eta^2 \left\{ \sum_i^{N-1} \frac{\mathbb{E}[n]}{(N-i)^{2\gamma}} \right. \\ &\quad \left. + 2 C_1 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N-1} \frac{\mathbb{E}[\sqrt{n}]^2}{(N-i)^\gamma (N-j)^\gamma (j-i)^\alpha} \right\} \\ &\sim N^{2-\alpha-2\gamma} \text{ as } N \rightarrow \infty \\ &\sim N \text{ only if } \gamma \approx (1-\alpha)/2. \end{aligned}$$

- For the French stocks considered by Bouchaud et al., the exponent  $\alpha \approx 0.2$  so  $\gamma \approx 0.4$ .

# The shape of the order book

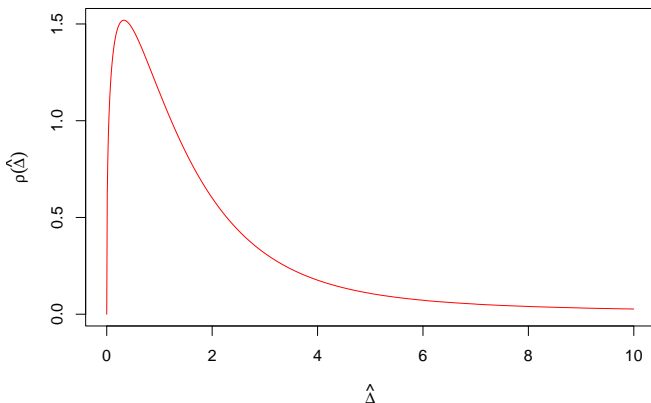
Bouchaud, Mézard and Potters (2002) derive the following approximation to the average density  $\rho(\hat{\Delta})$  of orders as a function of a rescaled distance  $\hat{\Delta}$  from the price level at which the order is placed to the current price:

$$\rho(\hat{\Delta}) = e^{-\hat{\Delta}} \int_0^{\hat{\Delta}} du \frac{\sinh(u)}{u^{1+\mu}} + \sinh(\hat{\Delta}) \int_{\hat{\Delta}}^{\infty} du \frac{e^{-u}}{u^{1+\mu}} \quad (9)$$

where  $\mu$  is the exponent in the empirical power-law distribution of new limit orders.

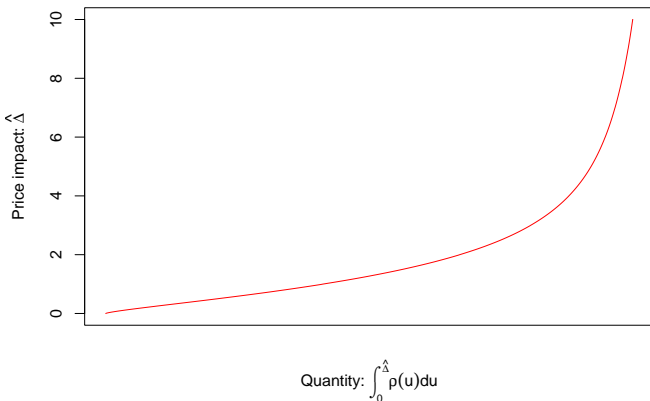
## Approximate order density

The red line is a plot of the order density  $\rho(\hat{\Delta})$  with  $\mu = 0.6$  (as estimated by Bouchaud, Mézard and Potters).



# Virtual price impact

Switching  $x$ - and  $y$ -axes in a plot of the cumulative order density gives the virtual impact function plotted below. The red line corresponds to  $\mu = 0.6$  as before.



## Impact for high trading rates

- You can't trade more than the total depth of the book so price impact increases without limit as  $n \rightarrow n_{max}$ .
- For a sufficiently large trading rate  $v$ , it can be shown that

$$f(v) \sim \frac{1}{(1 - v/v_{max})^{1/\mu}}$$

- Setting  $v = v_{max}(1 - \epsilon)$  and taking the limit  $\epsilon \rightarrow 0$ ,

$$f(v) \sim \frac{1}{\epsilon^{1/\mu}} \text{ as } \epsilon \rightarrow 0.$$

- Imagine we accumulate stock at a rate close to  $v_{max} := 1$  and liquidate at some (lower) rate  $v$ .
  - This is the pump and dump strategy!

## Impact for high trading rates continued

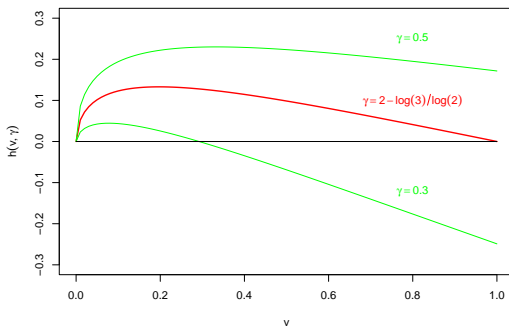
- Substituting into condition (7) gives

$$\frac{1}{\epsilon^{1/\mu}} \left\{ (1 - \epsilon) v^{1-\gamma} - (1 - \epsilon + v)^{2-\gamma} + (1 - \epsilon)^{2-\gamma} + v^{2-\gamma} \right\} + f(v) (1 - \epsilon)^{2-\gamma} \geq 0$$

- We observe that arbitrage is possible only if

$$h(v, \gamma) := v^{1-\gamma} - (1 + v)^{2-\gamma} + 1 + v^{2-\gamma} < 0$$

# Graphical illustration



- We see from the picture that  $h(v, \gamma) < 0$  is equivalent to

$$\gamma < \gamma^* := 2 - \frac{\log 3}{\log 2} \approx 0.415$$

- So if  $\gamma > \gamma^*$ , there is no arbitrage.

## More on high trading rates

- Also,  $h(v, \gamma)$  decreases as  $v \rightarrow v_{max}(= 1)$  so the arbitrage is maximized near  $v = v_{max}$ .
  - However, we already know that there is no arbitrage when trading in and out at the same rate.
- A careful limiting argument nevertheless shows that arbitrage is still possible in principle for every  $\gamma < \gamma^*$ .
- We deduce that, *independent* of the particular exponent  $\mu$  in the power law of limit order arrivals, the no-arbitrage condition is:

### Large size no arbitrage condition

$$\gamma > \gamma^* = 2 - \frac{\log 3}{\log 2}$$

# Summary

- Bouchaud et al. have previously noted that the market self-organizes in a subtle way such that the exponent  $\gamma$  of the power law of decay of market impact and the exponent  $\alpha$  of the decay of autocorrelation of trade signs balance to ensure diffusion (variance increasing linearly with time).

$$\gamma \approx (1 - \alpha)/2$$

- By imposing the principle of no-dynamic-arbitrage we showed that if the market impact function is of the form  $f(v) \propto v^\delta$ , we must have

$$\gamma + \delta \geq 1$$

- We excluded various other combinations of functional forms for market impact and decay such as exponential decay with nonlinear market impact.

## Summary continued

- We then observe that if the average cost of a (not-too-large) VWAP execution is roughly independent of duration, the exponent  $\delta$  of the power law of market impact should satisfy:

$$\delta + \gamma \approx 1$$

- By considering the tails of the limit-order book, we deduce that

$$\gamma \geq \gamma^* := 2 - \frac{\log 3}{\log 2} \approx 0.415$$

- Finally, we note that empirical estimates are  $\gamma \approx 0.4$  (Bouchaud et al.) and  $\delta \approx 0.6$  (Algren et al.)
- Our no-dynamic-arbitrage principle links these observations!

## Constraint on $\alpha$

- Assuming that autocorrelation of order signs is a long-memory process, we get

$$\gamma = \frac{1 - \alpha}{2}$$

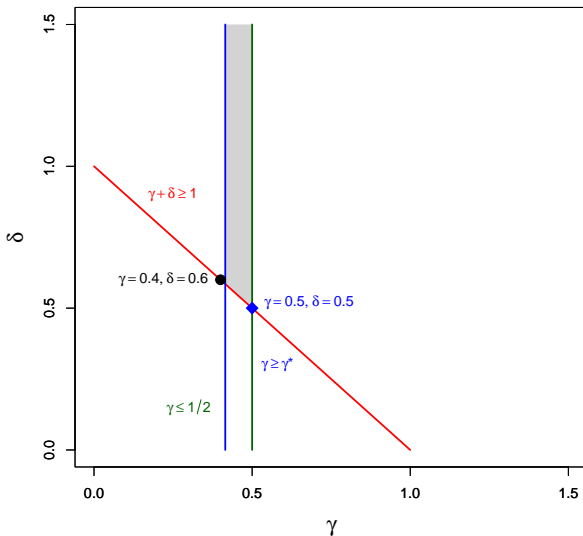
In particular, we must have  $\gamma \leq 1/2$ .

- Combining this with  $\gamma > \gamma^*$ , we obtain

$$\alpha \leq 1 - 2\gamma^* \approx 0.17$$

Faster decay is ruled out by no-dynamic-arbitrage.

# Schematic presentation of results



## More on the $\alpha$ constraint

- It seems that for most stocks,  $\alpha \approx 0.6$ .
- Bouchaud et al. and Farmer, Lillo et al. obtain similar estimates on average. For example, from Bouchaud, Kockelkoren and Potters (2006):

Stock	$\gamma$	$\alpha$
ACA	0.44	0.125
BN	0.26	0.61
CA	0.22	0.57
CGE	0.275	0.35
EN	0.27	0.57
EX	0.26	0.40
FP	0.15	0.69
FTE	0.44	0.14
GLE	0.13	0.58
MC	0.19	0.58
VIE	0.12	0.63

- This is inconsistent with the constraint  $\alpha < 0.17$ .
- Why?

## Potential explanations for violation of the constraint on $\alpha$ .

- “Pump and dump” is not a practical (or legal) arbitrage strategy.
- Order flow is not really a long-memory process.
- The assumed price process (1) is too unsophisticated.
  - For example, we assumed factorization of size and time effects; decay might depend on size.

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