

# Further Developments in Volatility Derivatives Modeling

Jim Gatheral



Global Derivatives Trading & Risk Management, Paris  
May 21, 2008

# Disclaimer

The opinions expressed in this presentation are those of the author alone, and do not necessarily reflect the views of of Merrill Lynch, its subsidiaries or affiliates.

# Motivation and context

- We would like to have a model that prices consistently
  - ① options on SPX
  - ② options on VIX
  - ③ options on realized variance
- We believe there may be such a model because we can identify relationships between options on SPX, VIX and variance. For example:
  - ① Puts on SPX and calls on VIX both protect against market dislocations.
  - ② Bruno Dupire constructs an upper bound on the price of options on variance from the prices of index options.
  - ③ The underlying of VIX options is the square-root of a forward-starting variance swap.
- The aim is not necessarily to find new relationships; the aim is to devise a tool for efficient determination of relative value.

# Outline

- ① Review of prior work
  - Double Lognormal vs Double Heston
- ② Parameter estimation
  - Time series analysis of variance curves
  - Time series analysis of SABR fits to SPX options
- ③ Model vs market prices
  - Numerical techniques
  - Pricing of options on VIX and SPX
  - Options on realized variance
- ④ Conclusion
  - Is the model right?

## Review of prior work

- We found that double-mean reverting lognormal dynamics for SPX instantaneous variance gave reasonable fits to both SPX and VIX option volatility smiles.
- Heston-style dynamics are inconsistent with option prices.
  - ① Double Lognormal fits better the Double-Heston on any given day.
  - ② Double Lognormal parameters are much more stable over time than Double Heston parameters.
- From time series of volatility factors extracted from variance curves, we found that:
  - ① Implied and historical volatility of volatility are similar
  - ② Implied mean reversion rates are lower than historical mean reversion rates
- We thus resolve some of the theoretical problems with one-factor stochastic volatility models.

## Double mean-reverting dynamics

- We consider double mean reverting dynamics of the form:

$$\begin{aligned}\frac{dS}{S} &= \sqrt{v} dW \\ dv &= -\kappa(v - v') dt + \xi_1 v^\alpha dZ_1 \\ dv' &= -c(v' - z_3) dt + \xi_2 v'^\beta dZ_2\end{aligned}\tag{1}$$

with  $\alpha, \beta \in [1/2, 1]$ .

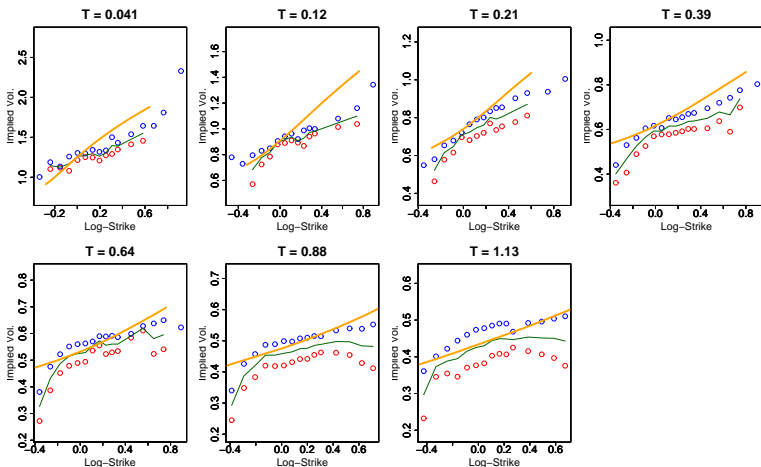
- We will call the case  $\alpha = \beta = 1/2$  *Double Heston*,
- the case  $\alpha = \beta = 1$  *Double Lognormal*,
- and the general case *Double CEV*.
- All such models involve a short term variance level  $v$  that reverts to a moving level  $v'$  at rate  $\kappa$ .  $v'$  reverts to the long-term level  $z_3$  at the slower rate  $c < \kappa$ .
- Initial conditions are  $v(0) = z_1$  and  $v'(0) = z_2$ .

# Fit of Double Lognormal model to VIX options

As of 03-Apr-2007, from Monte Carlo simulation with parameters

$$z_1 = 0.0137; z_2 = 0.0208; z_3 = 0.0421; \kappa = 12; \xi_1 = 7; c = 0.34; \xi_2 = 0.94;$$

we get the following fits (orange lines):

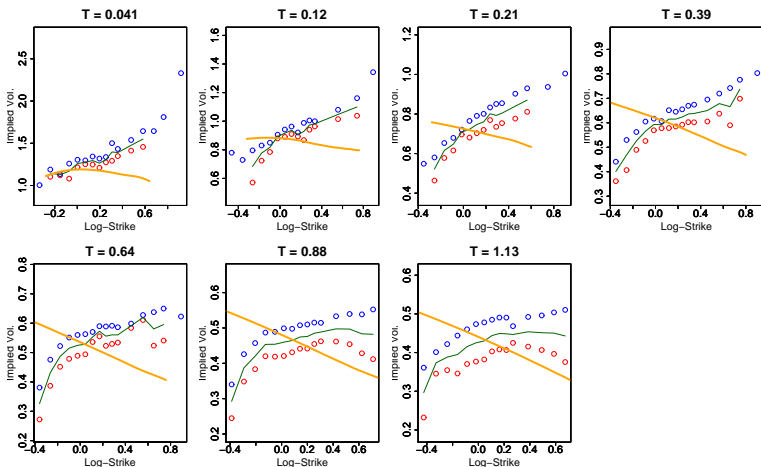


# Fit of Double Heston model to VIX options

As of 03-Apr-2007, from Monte Carlo simulation with parameters

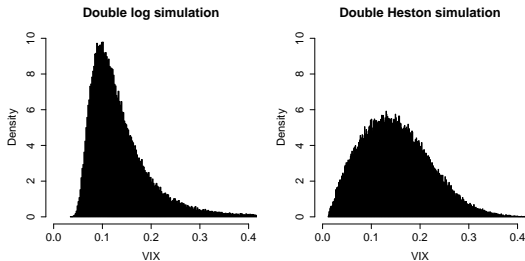
$$z_1 = 0.0137; z_2 = 0.0208; z_3 = 0.0421; \kappa = 12; \xi_1 = 0.7; c = 0.34; \xi_2 = 0.14;$$

we get the following fits (orange lines):



# In terms of densities of VIX

- When we draw the densities of VIX for the last expiration ( $T = 1.13$ ) under each of the two modeling assumptions, we see what's happening:



- In the (double) Heston model,  $v_t$  spends too much time in the neighborhood of  $v = 0$  and too little time at high volatilities.

## Parameter stability

- Suppose we keep all the parameters unchanged from our 03-Apr-2007 fit. How do model prices compare with market prices at some later date?
- Recall the parameters:
  - Lognormal parameters:

$$\kappa = 12; \xi_1 = 7; c = 0.34; \xi_2 = 0.94;$$

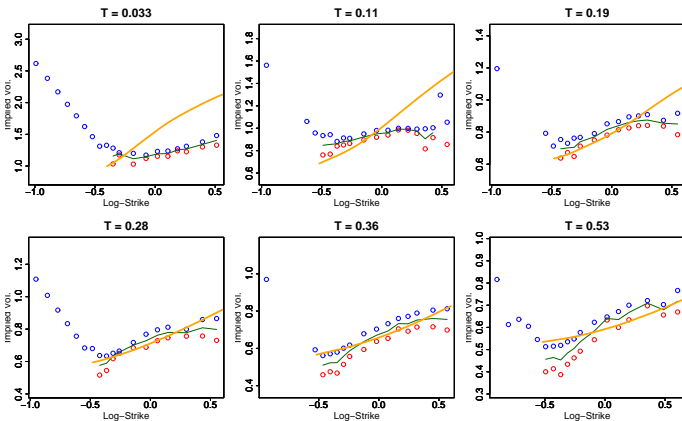
- Heston parameters:

$$\kappa = 12; \xi_1 = 0.7; c = 0.34; \xi_2 = 0.14;$$

- Specifically, consider 09-Nov-2007 when volatilities were much higher than April.
  - We re-use the parameters from our April fit, changing only the state variables  $z_1$  and  $z_2$ .

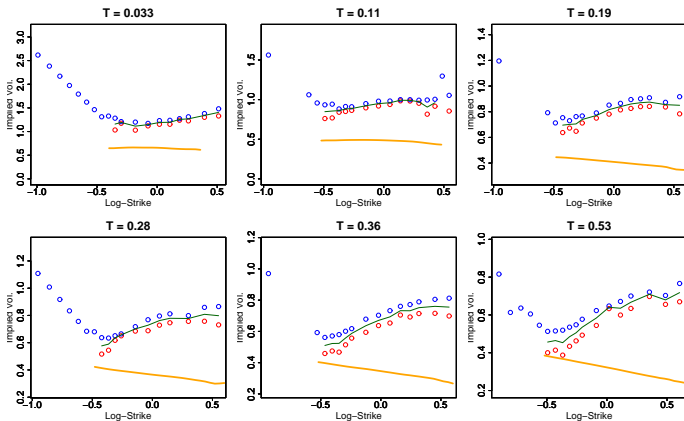
# Double Lognormal fit + VIX options as of 09-Nov-2007

With  $z_1 = 0.0745$ ,  $z_2 = 0.0819$  we get the following plots (model prices in orange):



# Double Heston fit + VIX options as of 09-Nov-2007

With  $z_1 = 0.0745$ ,  $z_2 = 0.0819$  we get the following plots (model prices in orange):

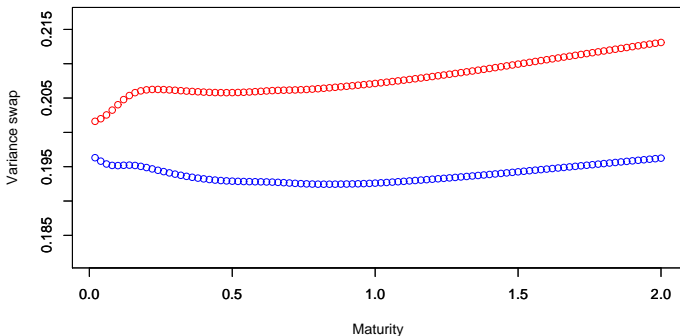


# Parameter estimation strategy

- Variance swaps don't depend on volatility of volatility
  - Analyze time series of variance swap curves to get  $\kappa$ ,  $c$  and  $z_3$ .
  - Get  $\rho$  and initial estimates of  $\xi_1$  and  $\xi_2$  from time series of factors.
- Fit SABR to SPX smiles to estimate the CEV exponent  $\alpha$ .
- Fit Double CEV to VIX options to deduce  $\xi_1$  and  $\xi_2$ .

## Average variance swap curves

Averaging the estimated variance curves over 7 years generates the following plot (SVI curve in red, ML curve in blue):



We note that the log-strip is only an approximation to the variance swap: interpolation and extrapolation methodology can make a big difference!

## Double CEV variance swap curve

- In the Double CEV model, given the state  $\{z_1, z_2\}$ , the variance swap curve is given by

$$z_3 + (z_1 - z_3) \frac{1 - e^{-\kappa T}}{\kappa T} + (z_2 - z_3) \frac{\kappa}{\kappa - c} \left\{ \frac{1 - e^{-c T}}{c T} - \frac{1 - e^{-\kappa T}}{\kappa T} \right\}$$

- One of the motivations for the Double CEV model is that it allows a richer set of potential variance swap curve shapes.

## Calibration of $\kappa$ , $c$ and $z_3$

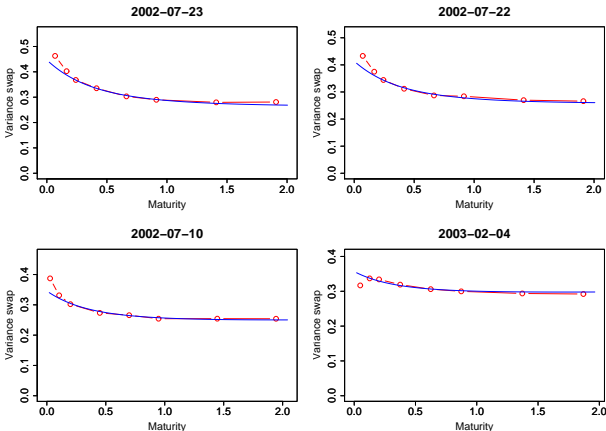
- We proceed as follows:
- For each day, and each choice of  $\kappa$ ,  $c$  and  $z_3$ 
  - 1 Impute  $z_1$  and  $z_2$  using linear regression
    - In the model, variance swaps are linear in  $z_1, z_2, z_3$
    - The coefficients are functions of  $T$  and the parameters  $\kappa$ ,  $c$
  - 2 Compute the squared fitting error
- Iterate on  $\kappa$ ,  $c$  and  $z_3$  to minimize the sum of squared errors
- Optimization results are:

Parameterization	$\kappa$	$c$	$z_3$
SVI	4.874	0.110	0.082
ML	5.522	0.097	0.074

- Processes have half-lives of roughly 7 weeks and 7 years respectively. Parameters are not too different from Bergomi's.

## Four worst fits

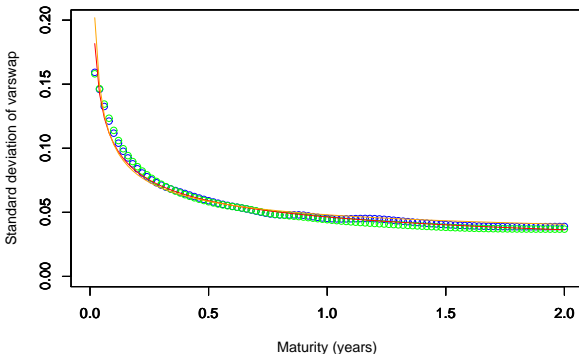
- The four worst individual SVI fits were as follows:



- We see real structure in the variance curve that the fit is not resolving.

# Volatility envelope

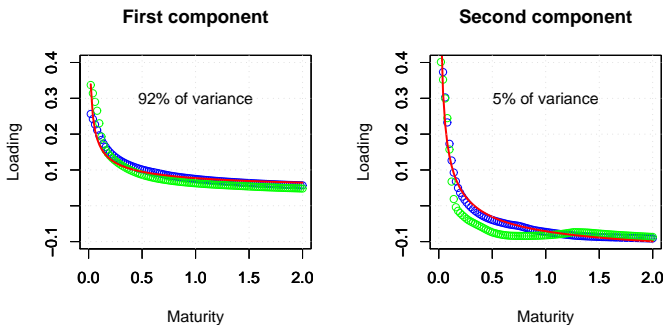
For each maturity, we compute the standard deviation of log-differences of the curves. ML and SVI results are green and blue respectively. The red and orange lines are proportional to  $1/\sqrt{T}$  and  $1/T^{0.36}$  respectively.



$\frac{1}{\sqrt{T}}$  seems to be a good approximation to the term structure of the volatility envelope!

# PCA on variance swap curves

- Perform PCA on log-differences of the SVI curves to obtain the following two factors:



- The blue and green points are from conventional and robust PCA respectively. The red lines are fits of the form  $a + b/\sqrt{T}$ .

## Motivation for fitting SABR

- It seems that volatility dynamics are roughly lognormal
  - Option prices and time series analysis lead us to the same conclusion.
- SABR is the simplest possible lognormal stochastic volatility model
  - And there is an accurate closed-form approximation to implied volatility.
- The lognormal SABR process is:

$$\begin{aligned}\frac{dS}{S} &= \Sigma dZ \\ \frac{d\Sigma}{\Sigma} &= \nu dW\end{aligned}\tag{2}$$

with  $\langle dZ, dW \rangle = \rho dT$ .

- As suggested by Balland, fitting SABR might allow us to impute effective parameters for a more complicated model (such as Double Lognormal).

## The SABR formula

As shown originally by Hagan et al., for extremely short expirations, the solution to (2) in terms of the Black-Scholes implied volatility  $\sigma_{BS}$  is approximated by:

$$\sigma_{BS}(k) = \sigma_0 f\left(\frac{k}{\sigma_0}\right)$$

where  $k := \log(K/F)$  is the log-strike and

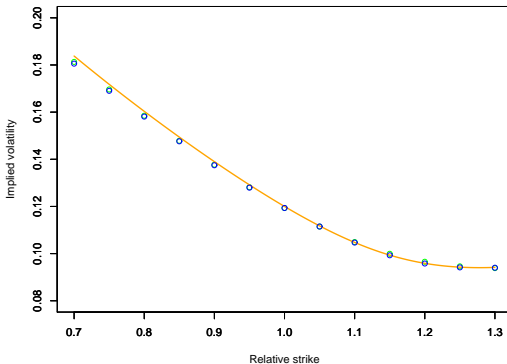
$$f(y) = -\frac{\nu y}{\log\left(\frac{\sqrt{\nu^2 y^2 + 2\rho\nu y + 1} - \nu y - \rho}{1 - \rho}\right)}$$

It turns out that this simple formula is reasonably accurate for longer expirations too.

- Note that the formula is independent of time to expiration  $T$

## How accurate is the SABR formula?

With  $T = 1$  and SABR parameters  $\nu = 0.5$ ,  $\rho = -0.7$  and  $\sigma_0 = 0.12$ , the following plot compares the analytical approximation with Monte Carlo and numerical PDE computations (in green and blue respectively):



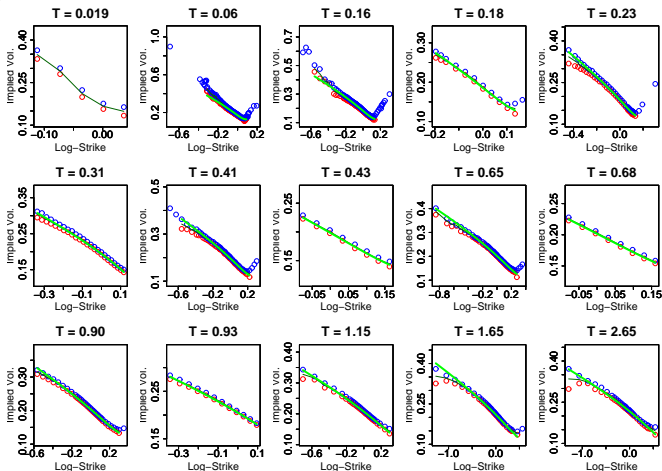
The formula looks pretty good, even for 1 year!

## Fitting SABR to SPX implied volatilities

- Consider the SPX option market on a given day (25-Apr-2008 for example).
- We fit the lognormal SABR formula to each timeslice of the volatility surface.
  - Then for each expiry, we impute  $\nu$ ,  $\rho$ ,  $\sigma_0$ .

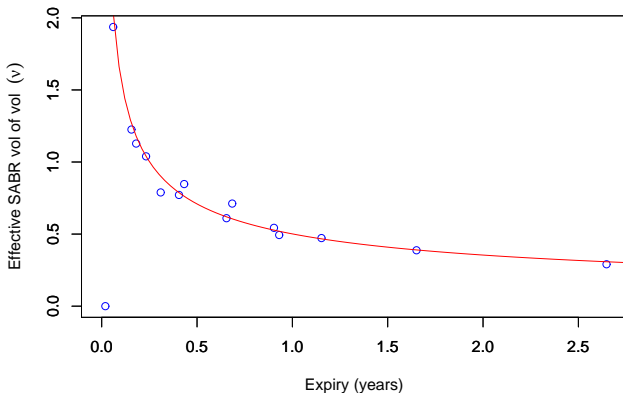
# Empirical and fitted volatility smiles

As of 25-Apr-2008, we obtain the following fits (SABR fits in green):



## The term structure of $\nu$

As of 25-Apr-2008, plot  $\nu$  for each slice against  $T_{exp}$ :



The red line is the function  $\frac{0.501}{\sqrt{T}}$ .

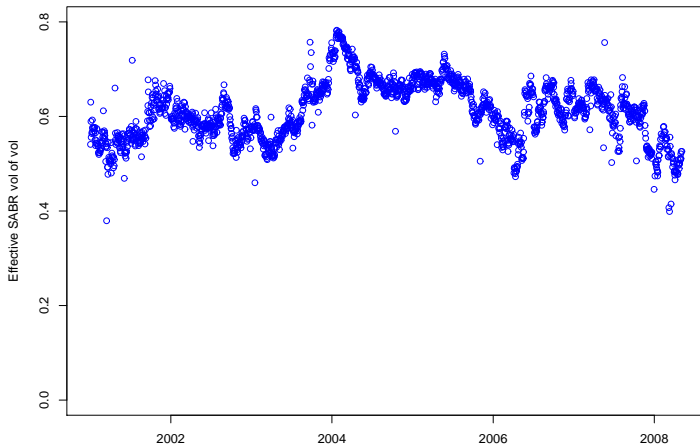
## Empirical observations

- We see that the term structure of  $\nu$  is almost perfectly  $1/\sqrt{T}$ .
  - Consistent with the empirical term structure of standard deviation of variance swaps
- This is found to hold for every day in the dataset.
- We can then parameterize the volatility of volatility on any given day by a single number:  $\nu_{eff}$  such that

$$\nu(T) = \frac{\nu_{eff}}{\sqrt{T}}$$

# SABR fits to SPX: $\nu_{eff}$

Computing  $\nu_{eff}$  every day for seven years gives the following time-series plot:

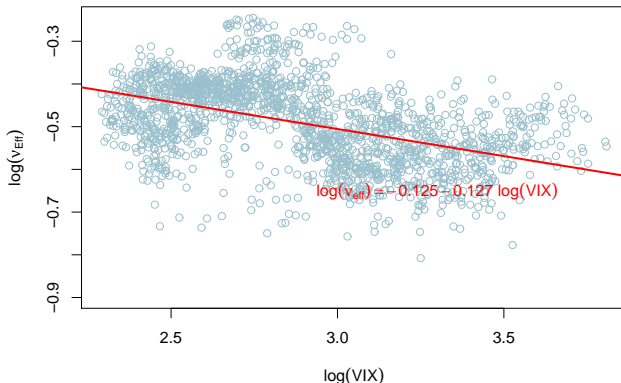


## Observations from $\nu_{eff}$ time-series

- Lognormal volatility of volatility  $\nu_{eff}$  is empirically rather stable
  - The dynamics of the volatility surface imply that volatility is roughly lognormal.
- Can we see any patterns in the plot?
  - For example, does  $\nu_{eff}$  depend on the level of volatility?

# Regression of $\nu_{eff}$ vs $VIX$

Regression does show a pattern!



$VIX \sim \sqrt{v}$  so we conclude that  $dv \sim v^{0.94} dZ$ .<sup>1</sup>

<sup>1</sup>The exponent of 0.94 coincides with an estimate of Aït-Sahalia and Kimmel from analysis of the VIX time series

# Parameters

- We finally settle on the following set of parameters:

Parameter	Value
$\kappa$	5.50
$c$	0.10
$z_3$	0.078
$\xi_1$	2.6
$\alpha$	0.94
$\xi_2$	0.45
$\beta$	0.94
$\rho$	0.57
$\rho_1$	-0.90
$\rho_2$	-0.70

- Let's call this the *final parameter set*.

# Monte Carlo

- Monte Carlo simulations of stochastic volatility models suffer from bias because even if variance remains positive in the continuous process, discretized variance may be negative.
- Various schemes have been suggested increase the efficiency of simulation of such models. For example:
  - Andersen (2006)
  - Lord, Koekkoek and Van Dijk (2008) (LKV)
- Given known moments, Andersen implements an Euler scheme for a different variance process that cannot go negative and whose moments match the first few of the known moments.
- The LKV approach is to slightly amend the Euler discretization at the boundary  $v = 0$ .
  - Since we don't have closed-form moments in general, we adopt a bias-corrected Euler scheme of the sort described in LKV.

## Monte Carlo discretization

We implement the following discretization of the Double CEV process (1):

$$v'_{t+\Delta t} = v'_t - c(v'_t - z_3)\Delta t + \xi_2 v'^{+\beta} \sqrt{\Delta t} Z_2$$

$$v_{t+\Delta t} = v_t - \kappa(v_t - v'_t)\Delta t + \xi_1 v^{+\alpha} \sqrt{\Delta t} Z_1$$

$$x_{t+\Delta t} = -\frac{1}{4}(v_t + v_{t+\Delta t})\Delta t + \sqrt{v^+} \sqrt{\Delta t} \{\rho_2 Z_2 + \phi_v Z_1 + \phi_x W\}$$

with  $\langle Z_1, Z_2 \rangle = \rho$ ;  $\langle Z_i, W \rangle = \rho_i$  ( $i = 1, 2$ ),  $x := \log S/S_0$  and

$$\phi_v = \frac{\rho_1 - \rho \rho_2}{\sqrt{1 - \rho^2}}$$

$$\phi_x = \sqrt{1 - \rho_2^2 - \phi_v^2}$$

This discretization scheme would be classified as “partial truncation” by LKV.

## Numerical PDE solution

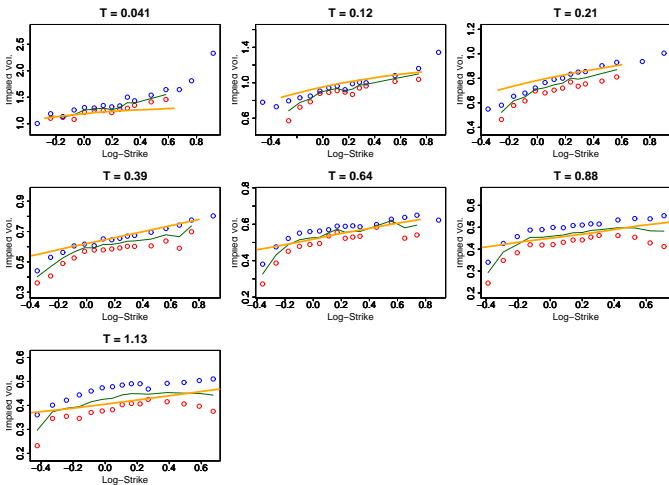
- The drift term of the Double CEV process is linear in  $v$  and  $v'$ , so  $VIX^2$  is linear in  $v$  and  $v'$  at the option expiration.
- If we had the joint distributions of  $v$  and  $v'$ , we would also have the distributions of  $VIX$  and all the option prices.
- This suggests trying to solve the Fokker-Planck (forward equation).
- We haven't succeeded in making such a scheme work so instead, we solve the backward equation for each strike and expiration.
  - We solve this equation using an ADI scheme.

## Numerical PDE vs Monte Carlo

- Although the numerical PDE solution is faster than Monte Carlo for a given accuracy, the code needs to be called once for each option.
- Monte Carlo can generate the entire joint distribution of  $x$ ,  $v$  and  $v'$  for each expiration.
  - With these joint distributions, we can price any option we want, including options on SPX and exotics.
- Implementation of a 3-dimensional numerical PDE solution is hard and the resulting code would be slow.
  - It's only practical to price options on VIX with numerical PDE.
- Accordingly, we use Monte Carlo in practice.

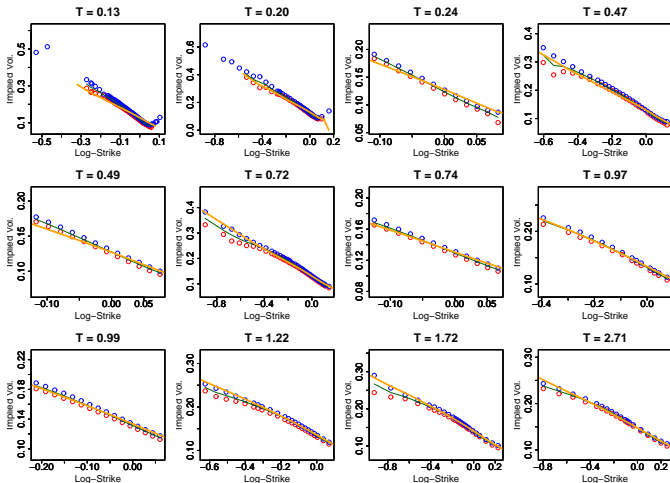
# Double CEV fit to VIX options as of 03-Apr-2007

From Monte Carlo simulation with the final parameter set, we get the following fits to VIX options (orange lines):



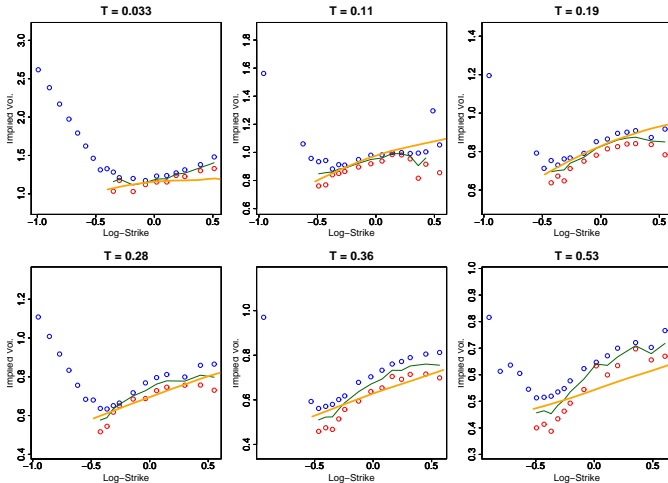
# Double CEV fit to SPX options as of 03-Apr-2007

Again from Monte Carlo simulation with the same parameters, we get the following fits to SPX options (orange lines):



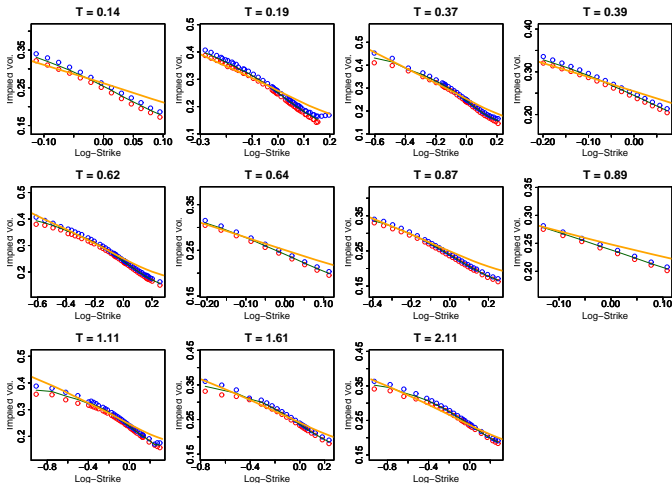
# Double CEV fit to VIX options as of 09-Nov-2007

From Monte Carlo simulation with our final parameters, we get the following fits to VIX options (orange lines):



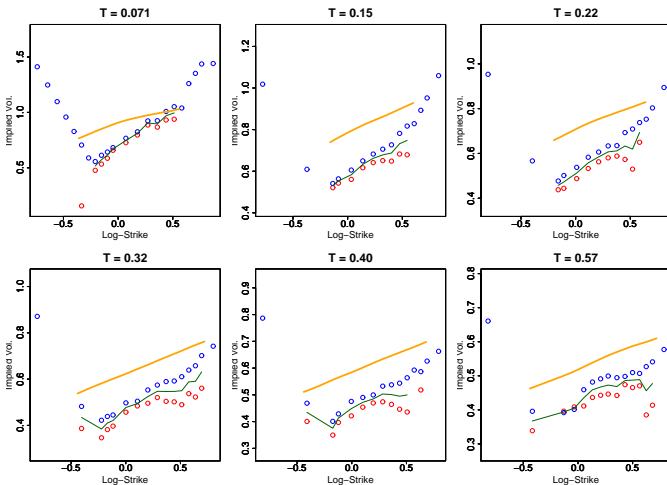
# Double CEV fit to SPX options as of 09-Nov-2007

Again from Monte Carlo simulation with our final parameters, we get the following fits to SPX options (orange lines):



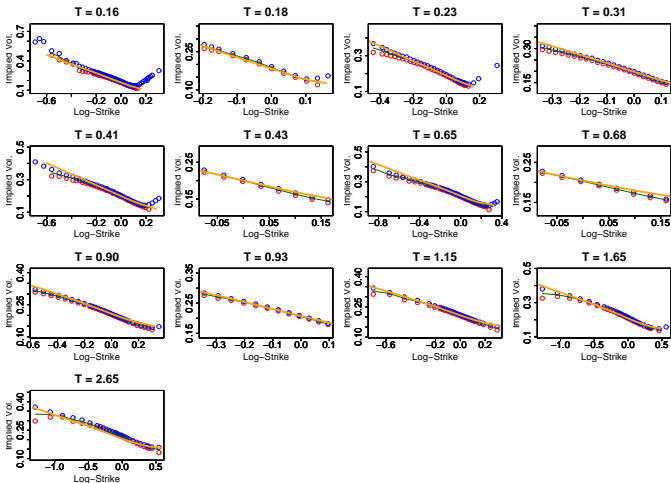
# Double CEV fit to VIX options as of 25-Apr-2008

From Monte Carlo simulation with our final parameters, we get the following fits to VIX options (orange lines):



# Double CEV fit to SPX options as of 25-Apr-2008

Again from Monte Carlo simulation with our final parameters, we get the following fits to SPX options (orange lines):



# Is volatility of volatility stable?

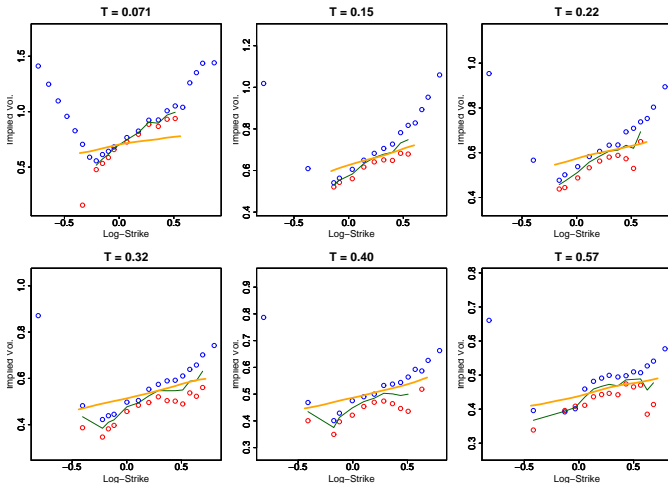
- Is volatility of volatility stable?
  - Of course not!
- Referring back to our SABR fits, we find:

Date	$\nu_{eff}$
03-Apr-2007	0.68
09-Nov-2007	0.61
03-Apr-2007	0.50

- So volatility of volatility decreased!

# Double CEV fit to VIX options as of 25-Apr-2008

From Monte Carlo simulation with proportionally lower vol-of-vol parameters, we get the following revised fits to VIX options (orange lines):



## How options on variance are quoted

- Define the realized variance as:

$$RV_T := \sum_i^T \Delta X_i^2$$

with  $\Delta X_i = \log(S_i/S_{i-1})$ .

- The price of an option on variance is quoted as

$$C = \frac{1}{2\sigma_K} \mathbb{E}[\text{payoff}]$$

where  $\sigma_K$  is the strike volatility.

- The price is effectively expressed in terms of volatility points on a variance swap quote. So if the quoted price of a call on variance is 2% and the strike price is 20%, the premium is  $2 \times 0.2 \times 0.02 = 0.008$  and the payoff is

$$\left( \frac{RV_T}{T} - \sigma_K^2 \right)^+$$

## Realized variance vs quadratic variation

- Option on variance are options on realized variance  $RV_T$ .
- In a model, we typically compute the value of an option on the quadratic variation  $QV_T$  defined as

$$QV_T := \int_0^T v_s ds$$

- Although  $\mathbb{E}[RV_T] = \mathbb{E}[QV_T]$  in a diffusion model,

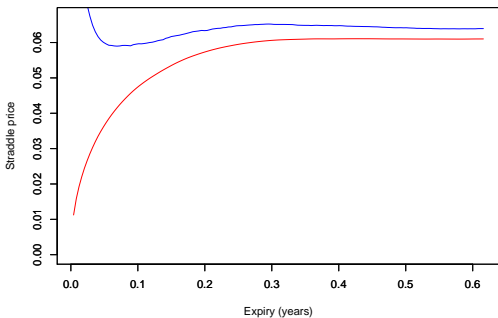
$$\text{Var}[RV_T] > \text{Var}[QV_T]$$

so options on  $RV$  are worth more than options on  $QV$ . One can think of  $RV_T$  as an approximation to  $QV_T$  with discretization error.

- It turns out that this discreteness adjustment is significant for shorter-dated options (under 3 months).

## Magnitude of discreteness effect

- Double CEV model prices of QV and RV straddles (in volatility points) as a function of the number of settings. QV is in red and RV in blue. Parameters are from November 2007.



- The discreteness effect is significant!

## Some broker quotes

- For the three dates for which we have computed model prices (03-Apr-2007, 09-Nov-2007 and 25-Apr-2008), we snap some broker prices of ATM variance straddles and compare our model prices.

Date	Expiry	Bid	Ask	Model	Model Adj.
03-Apr-2007	Jun-07	4.35	4.55	3.59	3.59
05-Apr-2007	Sep-07	3.90	4.70	3.74	3.74
07-Nov-2007	Jan-08	7.20	8.20	6.93	6.34
07-Nov-2007	Mar-08	4.35	7.20	7.08	6.49
13-Nov-2007	Jun-08	7.00	9.00	6.93	6.39
25-Apr-2008	Sep-08	5.60	6.10	5.25	4.20

## Some attributes of a good model

- 1 Must generate prices close to the market
- 2 Must have reasonable dynamics
  - Future scenarios for market prices should be consistent with stylized facts
  - For example, skews should not be too different from current skews
- 3 Parameters should be easy to identify
  - There should be an easy way to estimate parameter values from market observables
- 4 Parameters should be stable over time
- 5 Vanilla option values should be fast to compute
  - This is needed for efficient calibration

## Model scorecard

We can compare the single-factor Heston model to the Double CEV model along these attributes:

Attribute	Heston	Double CEV
Fits the market	Bad	Good
Reasonable dynamics	Medium	Good
Parameter identification	Medium	Bad
Parameter stability	Bad	Good
Easy vanillas	Good	Bad

# Summary

- The Double CEV model appears to reproduce market prices reasonably well.
  - SPX options and VIX options are more or less consistently priced. Options on realized variance less well priced.
- We reconfirmed using PCA that two factors are necessary.
- From regression of effective SABR volatility of volatility against VIX, we conclude that the CEV exponent  $\alpha$  in the volatility process is 0.94.
- However:
  - It's not clear how to estimate mean reversion and volatility of volatility parameters independently.
  - Computation is too slow for effective calibration.
- We suspect that there is a better two-factor volatility model with power-law decay of volatility autocorrelation coefficients.

## More general comments

- Although 2 factors are required, the two factors found from PCA each have roughly  $1/\sqrt{T}$  term structures.
- If the factors have a power-law structure, there is no particular timescale associated with them.
  - The timescales we settled on approximate a  $1/\sqrt{T}$  volatility envelope.
  - Other parameter choices that approximate this  $1/\sqrt{T}$  pattern seem to work just as well.
- VIX smiles are consistent with a model that has tighter distributions of volatilities than Double CEV.
  - SPX smiles are also consistent with tighter distributions of volatilities.
- In short, the Double CEV model is ugly!

## Current and future research

- Investigate alternative dynamics with power-law decay of volatility autocorrelations.
- Add more tradable factors (allow the skew to vary for example)

# References



Yacine Aït-Sahalia and Robert Kimmel.

Maximum likelihood estimation of stochastic volatility models.  
*Journal of Financial Economics*, 83:413–452, 2007.



Leif B.G. Andersen.

Efficient simulation of the heston stochastic volatility model.  
Technical report, Banc of America Securities, December 2006.



Philippe Balland.

Forward smile.  
Presentation at Global Derivatives, Paris, 2006.



Lorenzo Bergomi.

Smile dynamics ii.  
*Risk*, 18:67–73, October 2005.



Jim Gatheral.

*The Volatility Surface: A Practitioner's Guide*.  
John Wiley and Sons, Hoboken, NJ, 2006.



P Hagan, D Kumar, A Lesniewski, and D Woodward.

Managing smile risk.  
*Wilmott Magazine September*, pages 84–108, 2002.



Roger Lord, R Emmert Koekoek, and Dick van Dijk.

A comparison of biased simulation schemes for stochastic volatility models.  
Technical report, Erasmus University Rotterdam, February 2008.