New York University
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G22.2112-001/G63.2043 Scientific Computing

Handout #8
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Assignment 7

Exercise 1

Problem 6.4, Page 279.

Exercise 2

(Extra) Suppose that $a(x), u(x)$ are sufficiently smooth. Denote by Δ_0 the standard central difference operator, namely, $\Delta_0 a(x) = a(x+h/2) - a(x-h/2)$. Show that
(a)

$$(a(x)u'(x))' = \frac{1}{h^2}\Delta_0(a(x)\Delta_0 u(x)) + O(h^2); \tag{1}$$

(b)

$$(a(x)u'(x))' = \frac{1}{2h^2}\{ [a(x+h)+a(x)]\cdot[u(x+h)-u(x)]-[a(x)+a(x-h)]\cdot[u(x)-u(x-h)] \}+O(h^2). \tag{2}$$

Exercise 3

Suppose that a, b, f are sufficiently smooth functions, and that a, b are also positive. Consider finite difference solution of the boundary value problem

$$\begin{aligned} -(a(x)u'(x))' + b(x)u(x) &= f(x), \quad x \in (0,1), & (3)\\ u'(0) &= 0, & (4)\\ u'(1) + \gamma u(1) &= \beta & (5)\end{aligned}$$

on the equispaced grid $x_j = jh$, $j = 0, 1, \ldots, m$, $h = 1/m$.
(a) Use (2) to write down second order finite difference approximation of the equation (3) at the interior grid points $x_j, j = 1, 2, \ldots, m-1$;
(b) Use the second order forward difference scheme

$$u'(x) = \frac{1}{h}[\Delta_+ - 0.5\Delta_+^2]u(x) + O(h^2); \quad \text{where } \Delta_+ u(x) = u(x+h) - u(x) \tag{6}$$

to construct approximation of the left boundary condition (4) (this is a finite difference equation for the first grid point x_0);
(c) Similarly use the second order backward difference approximation

$$u'(x) = \frac{1}{h}[\Delta_- + 0.5\Delta_-^2]u(x) + O(h^2); \quad \text{where } \Delta_- u(x) = u(x) - u(x-h) \tag{7}$$

to construct approximation of the right boundary condition (5) (this is a finite difference equation at the last grid point x_m);
(d) Given

$$\begin{aligned} a(x) &= 1 + x\ln(2+x), & (8)\\ b(x) &= 1/(2+x), & (9)\\ f(x) &= (1+x^2)(1+\cos(15x)) & (10)\\ \gamma &= 1, \ \beta = 2, & (11)\end{aligned}$$

solve this linear system of $m+1$ equations for $m+1$ unknowns $u_j, j = 0, 1, \ldots, m$ for $m = 40, 80, 160$. Check and show rate of convergence (note that the linear system is tridiagonal except at the first and the last rows);
(e) Plot the numerical solution in $[0, 1]$ for $m = 160$.

Exercise 4

Use the standard second order, five-point stencil, finite difference scheme to solve the Poisson equation

$$-\nabla^2 u(x,y) = f(x,y), \quad (x,y) \in [0,\pi] \times [0,\pi] \tag{12}$$

subject to the Dirichlet boundary conditions

$$\begin{aligned} u(0,y) &= g_l(y), \quad y \in [0,\pi], & (13)\\ u(\pi,y) &= g_r, \quad y \in [0,\pi], & (14)\\ u(x,0) &= g_b, \quad x \in [0,\pi], & (15)\\ u(x,\pi) &= g_t, \quad x \in [0,\pi]. & (16) \end{aligned}$$

where the subscripts l, r, b, t represent left, right, bottom, and top.
(a) **Read and understand this paragraph only, do nothing else.** Let's first consider the simplest case that is still general enough to see the structure of the discretization: $n = 5$, $h = \pi/n$. In this case, there are $6 \times 6 = 36$ mesh points all together: $(x_i,\ y_j) = (ih, jh)$, $0 \le i, j \le 5$. There are thus $4 \times 4 = 16$ interior mesh points, and $4 \times 5 = 20$ mesh points on the boundary. There are 16 unknowns

$$u_{ij} =: u(x_i, y_j), \quad 1 \le i, j \le 4 \tag{17}$$

and we want to re-organize them as a vector of length 16, or $(n-1)^2$

$$u^h = \begin{bmatrix} u_1^h \\ u_2^h \\ u_3^h \\ u_4^h \end{bmatrix}, \quad \text{where} \quad u_j^h = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ u_{4,j} \end{bmatrix}. \tag{18}$$

In other words, we have $u^h(k) = u_{ij}$ with $k = (n-1) \cdot (j-1) + i$. Note that the vector u^h is now indexed by a single integer k whereas the vector u is indexed by two integers (i, j); as i and j range from 1 to $n-1$, k ranges from 1 to $(n-1)^2$.
(b) **Read and understand this paragraph only, do nothing else.** The unknown vector u^h satisfies the linear system of equations with an error term $O(h^2)$

$$A\ u^h = f^h + g^h + O(h^2), \tag{19}$$

where the vector u^h still assumes the exact values of the function $u(x, y)$ at the $(n-1)^2$ interior mesh points. Abusing the notation slightly, we denote the approximate (numerical) solution also by u^h which satisfies the linear system of equations

$$A\ u^h = f^h + g^h \tag{20}$$

where

$$f^h(k) = f(x_i, y_j), \quad 1 \le i, j \le 4, \quad 1k = (n-1) \cdot (j-1) + i \tag{21}$$

(c) **You are now asked to do something.** What are the dimensions of $A,\ f^h,\ g^h$
(d) Partition A as 4-by-4 blocks (in general, (n-1)-by-(n-1) blocks)

$$A = \frac{1}{h^2} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}. \tag{22}$$

What are the dimensions of each of the blocks A_{ij}. Note that this is a block-tridiagonal matrix with

$$A_{ii} = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}, \; i = 1:4; \quad A_{ij} = -I_4, \; |i-j| = 1; \quad A_{ij} = 0 \quad \text{otherwise} \tag{23}$$

(e) Partition the vectors f^h, g^h as 4-by-1 blocks. What are the dimensions of the blocks. Determine the entries of each block.
(f) Implement the resulting linear system (20) on a computer for a general n value.
(g) For $n = 6,\ 12,\ 24$, solve for u^h with $f(x) = 0$ and subject to the Dirichlet boundary conditions

$$g_l(y) = \sin(\alpha y), \quad g_r = \sin(\beta y), \quad g_b(x) = g_t(x) = 0. \tag{24}$$

Choose $\alpha = 1$ and $\beta = 1$ to check order of convergence (it should be second order in h). Use $n = 12,\ 16,\ 22$ to check order if necessary.
(h) Use a suitable n value to produce a numerical solution u^h for $\alpha = 15$, $\beta = 7$. Make a surface plot of the corresponding vector $u_{ij} = u^h(k)$ with $k = (n-1)\cdot(j-1)+i$.
(i) Finally, for those with adequate exposure to PDEs, explain the behavior of the (numerical) solution.

Exercise 5

Optional. Under the conditions of Exercise 3, consider finite difference solution of the initial-boundary value problem for the diffusion equation

$$u(x,t) = (a(x)u'(x,t))' - b(x)u(x,t) + f(x), \quad x \in (0,1), \tag{25}$$

$$u(x,0) = \begin{cases} 1 - \cos(4\pi x), & x \in (0, 0.5) \\ 1 - \cos(8\pi x), & x \in (0.5, 1) \end{cases} \tag{26}$$

$$u'(0,t) = 0, \tag{27}$$

$$u'(1,t) + \gamma u(1,t) = \beta \tag{28}$$

Let $u_h = [u(x_0), u(x_1), \cdots, u(x_m)]'$ be the numerical solution of Exercise 3. Denote by $U_h(t) = [u(x_0,t), u(x_1,t), \cdots, u(x_m,t)]'$ be the numerical solution of Exercise 5. Let $Au_h = f_h$ be the discrete system obtained from Exercise 3 with the spatial grid $x_j = jh,\ j = 0, 1, \ldots, m,\ h = 1/m$. Then $U_h(t)$ satisfies the semi-discretized diffusion equation – a system of ODEs for the vector $U_h(t)$

$$\frac{d}{dt}U_h(t) = -A \cdot U_h(t) + f_h \tag{29}$$

to the second order of h.
(a) Solve the ODEs (29) with the forward Euler's method with $\Delta t = 2h^2$ and observe that the marching in t-steps is not stable.
(b) Same as above but with $\Delta t = 0.5h^2$ and see if it is stable. If not, try $\Delta t = 0.2h^2$.
(c) Solve the ODEs (29) with the backward Euler's method with $\Delta t = 2h^2$ and observe that the marching in t-steps is stable.
(d) Same as above but with $\Delta t = h$ and observe that the marching in t-steps is stable.
(e) Solve the ODEs (29) with the trapezoidal method with $\Delta t = h$. Check order of convergence at $t = 0.5$ with $m = 40, 80, 160$. It should be convergent with second order.
Note: Solve each case till $t = 2$. See (8)–(11) for the parameters. For the backward Euler and trapezoidal method, use sparse LU factorization for the matrix A once for a fixed m value, and then do the back solve for each t-step.
(f) Plot $U_h(t\ )$for $m = 80$ and $t = 2$ and compare it with the plot of Exercise 3.
(g) Comment on the efficiencies of the forward, backward Euler, and the trapezoidal methods.