Assignment 5

1. 2. 3. Problem 8.1, 8.3, 8.4; due April 5. The next problem due April 12

4. Denote by $J_0(z)$ the Bessel J function of order 0; in Matlab it is specified by `nu=0; bj=besselj(nu,z)`. Let

$$u_j(z) = c_j \cos(a_j z + b_j), \tag{1}$$

and let $a = [a_1, a_2, \ldots, a_m]'$ be a column vector. Define the column vectors b and c the same way. The purpose of this exercise is to approximate $J_0(z)$ in $[\alpha, \beta] = [0, 2\pi]$ by $u_j(z)$ for $j = 1 : m$ by solving the so-called (nonlinear) least-squares problem formulated as

$$\min\ F(a, b, c) =: \int_\alpha^\beta \left[J_0(z) - \sum_{j=1}^{m} u_j(z) \right]^2 dz, \tag{2}$$

which is an unconstrained nonlinear optimization problem. Here we'll consider only the cases of $m = 1$ and $m = 2$. We'll also replace the integral by a Gaussian quadrature of $n = 10$ nodes $\{z_i\}$ and weights $\{w_i\}$, scaled and translated to $[\alpha, \beta]$), so that the optimization problem (2) becomes

$$\min\ f(a, b, c) =: \sum_{i=1}^{n} w_i \left[J_0(z_i) - \sum_{j} u_j(z_i) \right]^2, \tag{3}$$

Remark. See Problem 4 of HW4 for the calculation of the roots $\{x_i\}$ of Legendre polynomial, then $z_i = x_i \cdot (\beta - \alpha)/2 + (\beta + \alpha)/2$ are the Gaussian nodes. The weights are given by the formula $w_i = 2/\{(1 - x_i^2)[p_n'(x_i)]^2\}$, see Abramowitz and Stegun, page 887, section 25.4.29, where p is the Legendre polynomial of degree n and with $p_n(1) = 1$ — the one we used in Problem 4 of HW4.

a. Plot the function $J_0(z)$ in $[\alpha, \beta]$.

b. For $m = 1$, only one function $u_1(z) = c_1 \cos(a_1 z + b_1)$ is used to approximate $J_0(z)$ in $[\alpha, \beta]$. Find $a_1,\ b_1,\ c_1$ to solve the optimization problem (3) by finding the zeros of the gradient of f. Namely, solve the three equations $\nabla f(a_1, b_1, c_1) = 0$ by Newton's iteration (denote a zero by $(\bar{a}_1, \bar{b}_1, \bar{c}_1)$, a point in 3-D)

c. Provide the residual f and the gradient ∇f at $(\bar{a}_1, \bar{b}_1, \bar{c}_1)$. Plot the error function

$$e_1(z) = J_0(z) - u_1(z). \tag{4}$$

d. How do you make sure that the zero you find corresponds the global minimum of the error function.

e. Extra. Fix $(a_1, b_1, c_1) = (\bar{a}_1, \bar{b}_1, \bar{c}_1)$ and repeat steps (b), (c) and (d) with $J_0(z)$ replaced by $e(z)$, $u_1(z)$ replaced by $u_2(z)$, and e_1 replaced by

$$e_2(z) = J_0(z) - u_1(z) - u_2(z). \tag{5}$$

Hint. For each optimization problem, it is important to find a good initial guess to start the Newton's iteration, particularly for a_1, and a_2 if you do the last part (e). Experiment by looking at the plots of $J_0(z)$ v.s. that of $u_1(z)$, and the plots of the error function (4).