An Error Estimate for Recursive Linearization of Inverse Scattering Problems

Gang Bao
Department of Mathematics
University of Florida
Gainesville, FL 32611-2080

Ma Fuming
Department of Mathematics
Jilin University
Changchun,130023
P.R.China

Abstract

In this note a recursive linearization procedure for solving inverse scattering problems is studied. An error estimate is established.

1 Introduction

Consider the nonlinear equation

$$F(q,t)=0, \quad q \in R^M, \quad t \in [0,1], \tag{1.1}$$

where F: $R^M \times [0,1] \to R^M$, t is a real parameter. Assume that F is differentialbe with respect to q and t. The setting arises naturally in the method of continuation. The goal is to determine a solution $q(1)$ of the equation

$$F(q,1)=0. \tag{1.2}$$

If the derivative $\frac{\partial F(q,t)}{\partial q}$ has a bounded inverse for $t \in [0,1]$, then the standard continuation procedure may be used to compute an approximate solution of (1.2) for a given approximation q_0 of the equation

$$F(q,0)=0. \tag{1.3}$$

Unfortunately, for discretized equations of type (1.1) resulting from inverse scattering problems[1]-[3], $\frac{\partial F(q,t)}{\partial q}$ is usually singular or nearly singular, due to the ill-posedness nature of this class of inverse problems. In this note, we obtain an error estimate for the recursive linearization procedure employed particularly in Chen[1][2] for solving inverse medium problems.

2 Recursive Linearization

Following Isaacson[4], Dobson[5], we make use of the concept indistinguishability, which is convenient in analyzing the uncertainty properties of inverse scattering problems[1].

Definition 2.1 *Two vectors $q_1, q_2 \in R^M$ are said to be indistinguishable by measurements of precision ϵ iff*

$$\|F(q_1, t) - F(q_2, t)\| \leq \epsilon \; .$$

Similarly one may define the set of indistinguishable perturbations for a map $G(q)$ by

$$G_\epsilon(q) = \{\delta q \in R^M, \|G(q)\delta q\| \leq \epsilon\} \; .$$

Denote by $P = P(q,t)$ the orthogonal projection operator from R^M to the invariant subspace corresponding to nonzero eigenvalues of $[\frac{\partial F(q,t)}{\partial q}]^*[\frac{\partial F(q,t)}{\partial q}]$. Let $Q = I - P$. It follows from Definition 2.1 that for $A(q) = \frac{\partial F(q,t)}{\partial q}$,

$$\text{Range } Q \subset A_\epsilon(q) \quad \forall \epsilon > 0$$

or

$$\frac{\partial F(q,t)}{\partial q}\delta q = 0, \quad \forall \delta q \in \text{Range} Q \; .$$

We are now ready to state a recursive linearization procedure:

Procedure for finding an approximation of $q(1)$.

Step 1. Initialize a discretization grid of [0,1]: $0 = t_0 < t_1 < \cdots < t_N = 1$ and an approximate solution q_0 of Equation (1.3).

Step 2. For $n = 0, 1, \cdots, N-1$, compute

$$q_{n+1} = q_n + \delta q_n \; , \tag{2.1}$$

where δq_n solves the least-squares problem

$$\min_{\delta q \in Range\ P(q_n, t_n)} \|\frac{\partial F(q_n, t_n)}{\partial q}\delta q_n + \frac{\partial F(q_n, t_n)}{\partial t}(t_{n+1} - t_n)\|^2. \tag{2.2}$$

3 An error estimate

We begin with some general hypotheses:

(H1) There exists a solution $\tilde{q}(t)$ of the equation (1.1) for any $t \in [0,1]$. Further $\tilde{q}(t)$ is twice differentiable and

$$\|\tilde{q}(t)\|, \quad \|\tilde{q}'(t)\|, \quad \|\tilde{q}''(t)\| \leq C, \quad \forall t \in [0,1].$$

(H2) The derivatives $\frac{\partial F}{\partial q}$, $\frac{\partial F}{\partial t}$ are P-Lipschitz continuous in the following sense

$$\|\frac{\partial F}{\partial q}(q,t) - \frac{\partial F}{\partial q}(\tilde{q}(t),t)\| \leq C\|P(\tilde{q}(t),t)(q-\tilde{q}(t))\|$$

for any $q \in R^M$ and $t \in [0,1]$, where again $P(\tilde{q}(t),t)$ is the projection operator defined in Section 2. Similarly

$$\|\frac{\partial F}{\partial t}(q,t) - \frac{\partial F}{\partial t}(\tilde{q}(t),t)\| \leq \|P(\tilde{q}(t),t)(q-\tilde{q}(t))\|$$

for any $q \in R^M$ and $t \in [0,1]$.

(H3) The operator defined by

$$A(q,t) = P^*(q,t)[\frac{\partial F}{\partial q}(q,t)]^*[\frac{\partial F}{\partial q}(q,t)]P(q,t)$$

has a bounded inverse uniformly with respect to q, t.

Let

$$\tilde{q}(0), \quad \tilde{q}(t_1), \quad \cdots, \quad \tilde{q}(t_N) = \tilde{q}(1)$$

be the exact solutions of Equation (1.1). Assume that

$$q_0, \quad q_1, \quad \cdots, \quad q_N$$

is a sequence of approximate solutions generated by the recursive linearization procedure of Section 2.

Set

$$e_n = \tilde{q}(t_n) - q_n, \quad \Delta t_n = t_{n+1} - t_n.$$

We then have from Taylor's theorem and (2.1) that

$$\begin{aligned} e_{n+1} &= \tilde{q}(t_{n+1}) - q_{n+1} \\ &= \tilde{q}(t_n) - q_n + \tilde{q}'(t_n)\Delta t_n - \delta q_n + O(\Delta t_n^2) \\ &= e_n + \tilde{q}'(t_n)\Delta t_n - \delta q_n + O(\Delta t_n^2). \end{aligned} \tag{3.1}$$

From (2.2), we get the normal equation

$$[\frac{\partial F}{\partial q}(q_n,t_n)]^*[\frac{\partial F}{\partial q}(q_n,t_n)]\delta q_n = -[\frac{\partial F}{\partial q}(q_n,t_n)]^*[\frac{\partial F}{\partial t}(q_n,t_n)]\Delta t_n \ , \tag{3.2}$$

where

$$\delta q_n \in \text{Range } P(q_n, t_n) \ .$$

It follows from Hypothesis (H3) that

$$\|\delta q_n\| \leq C\Delta t_n \ . \tag{3.3}$$

Thus, we obtain

Lemma 3.1 *Under the hypotheses (H1)-(H3), there is a constant C, such that*

$$\|e_n\| \leq C, \quad \text{for} \quad 0 \leq n \leq N,$$

where the constant is independent of N, $\Delta t = \max_n \Delta t_n$.

Our next two lemmas are concerned with continuity properties of the projection operator P.

Lemma 3.2 *Under the hypotheses (H1), (H2), the following estimate holds*

$$\|P(q,t) - P(\tilde{q}(t),t)\| \leq C\|P(\tilde{q}(t),t)(q - \tilde{q}(t))\|$$

for any q *and* $t \in [0,1]$.

Proof. Set

$$B(q,t) = [\frac{\partial F}{\partial q}(q,t)]^*[\frac{\partial F}{\partial q}(q,t)].$$

From the hypotheses (H1) (H2), we have

$$\|B(q,t) - B(\tilde{q}(t),t)\| \leq C\|P(\tilde{q}(t),t)(q - \tilde{q}(t))\| \ .$$

The proof can then be completed by using the formula on singularity of the resolvent (see Kato[6], p.p. 38-39)

$$P(q,t) = -\frac{1}{2\pi i}\int_\Gamma (B(q,t) - \lambda I)^{-1} d\lambda \ ,$$

where Γ is a positively-oriented circle in the complex plane which encloses all nonzero eigenvalues of B but excludes zeros. $\square$

For simplicity, let us denote $P(\tilde{q}(t_n), t_n)$ by P_n.

Lemma 3.3 *Under the hypotheses (H1)-(H3), the following estimate holds*

$$\|P_{n+1} - P_n\| \leq C\Delta t_n, \quad n = 0, 1, \cdots, N-1 \;.$$

The proof is straight-forward by using the hypotheses and Lemma 3.2.

Lemma 3.4 *Under the hypotheses (H1)-(H3), there is a constant C, such that*

$$\|P_n\tilde{q}'(t_n)\Delta t_n - \delta q_n\| \leq C\|P_n e_n\|\Delta t_n \;.$$

Proof. Denote

$$\hat{P}_n = P(q_n, t_n), \quad A = A(\tilde{q}(t_n), t_n), \quad \hat{A} = A(q_n, t_n) \;.$$

We then have

$$AP_n\tilde{q}'(t_n) = -P_n^* f$$

and

$$\hat{A}\delta q_n = -\hat{P}_n^*\hat{f}\Delta t_n \;,$$

where

$$f = [\frac{\partial F}{\partial q}(\tilde{q}(t_n), t_n)]^* \frac{\partial F}{\partial t}(\tilde{q}(t_n), t_n)$$

and

$$\hat{f} = [\frac{\partial F}{\partial q}(q_n, t_n)]^* \frac{\partial F}{\partial t}(q_n, t_n) \;.$$

Thus

$$\begin{aligned} P_n\tilde{q}'(t_n)\Delta t_n - \delta q_n &= (\hat{A}^{-1}\hat{P}_n^*\hat{f} - A^{-1}P_n^* f)\Delta t_n \\ &= (\hat{A}^{-1}\hat{P}_n^* - A^{-1}P_n^*)\hat{f}\Delta t_n + A^{-1}P_n^*(\hat{f} - f)\Delta t_n \;. \end{aligned} \tag{3.4}$$

Let us first estimate the second term of the right hand side of (3.4). Using the hypotheses (H1)-(H3), it is easily seen that

$$\|A^{-1}P_n^*(\hat{f} - f)\Delta t_n\| \leq C\|P_n e_n\|\Delta t_n \;.$$

Next let

$$H = \mathrm{Range}Q_n^* + \mathrm{Range}\hat{Q}_n^* \;.$$

Clearly H is a subspace of R^M. Further, for any vector $\hat{f} \in R^M$,

$$\hat{f} = \hat{f}_1 + \hat{f}_2, \quad \hat{f}_1 \in H \;, \quad \hat{f}_2 \in H^{\perp}.$$

Since $\hat{f}_1 \in H$, we also have

$$\hat{f}_1 = \hat{f}_1' + \hat{f}_1'', \quad \hat{f}_1' \in Range\ Q_n^*, \hat{f}_1'' \in Range\ \hat{Q}_n^*.$$

Combining the above, we have

$$\hat{f}_1 = (I - P_n^*)\hat{f}_1' + (I - \hat{P}_n^*)\hat{f}_1''.$$

In addition,

$$\hat{f}_2 \in Range\ P_n^* \bigcap Range\ \hat{P}_n^* ,$$

since

$$\hat{f}_2 \perp Range\ Q_n^*, \quad \hat{f}_2 \perp Range\ \hat{Q}_n^* .$$

Therefore

$$\begin{aligned}
& \|(\hat{A}^{-1}\hat{P}_n^* - A^{-1}P_n^*)\hat{f}\| \\
\leq\ & \|(\hat{A}^{-1}\hat{P}_n^* - A^{-1}P_n^*)\hat{f}_2\| + \|\hat{A}^{-1}\hat{P}_n^*(I - P_n^*)\hat{f}_1'\| + \|A^{-1}P_n^*(I - \hat{P}_n^*)\hat{f}_1''\| \\
\leq\ & \|(\hat{A}^{-1} - A^{-1})\hat{f}_2\| + C\|\hat{P}_n^*(I - P_n^*)\hat{f}_1'\| + C\|P_n^*(I - \hat{P}_n^*)\hat{f}_1''\| \\
\leq\ & C\|P_n e_n\| , \qquad (3.5)
\end{aligned}$$

where in order to get the above estimate, we have used the boundedness of $\hat{f}$ and

$$\|P_n^*(I - \hat{P}_n^*)\| = \|(I - \hat{P}_n)P_n\|,$$

$$\|\hat{P}_n^*(I - P_n^*)\| = \|(I - P_n)\hat{P}_n\|.$$

Finally, the proof is completed by combining (3.4) and (3.5). □

We can now state and prove the main result of this note.

Theorem 3.1 *Under the hypotheses (H1)-(H3), the following error estimate holds*

$$\|P_n e_n\| \leq C(\|P_0 e_0\| + \Delta t + \max_{0 \leq l \leq n-1} \frac{1}{\Delta t_l}\|P_{l+1}Q_l\|) , \qquad (3.6)$$

where C is a constant independent of n and $\Delta t = \max_{0 \leq n \leq N-1} \Delta t_n$. In particular

$$\|P_N(q_N - q(1))\| \leq C(\|P_0(q_0 - q(0))\| + \Delta t + \max_{0 \leq n \leq N-1} \frac{1}{\Delta t_n}\|P_{n+1}Q_n\|) . \qquad (3.7)$$

Proof. From (3.1), we have

$$\begin{aligned}
P_{n+1}e_{n+1} \ =\ & (P_{n+1} - P_n)(e_n + \tilde{q}'(t_n)\Delta t_n - \delta q_n) + P_n e_n \\
& + P_n \tilde{q}'(t_n)\Delta t_n - \delta q_n + (\hat{P}_n - P_n)\delta q_n + O(\Delta t_n^2) .
\end{aligned}$$

It follows from Lemmas 3.1-3.3 and the equation (3.2) that

$$\|P_{n+1}e_{n+1}\| \le \|P_n e_n\| + C\Delta t_n \|P_n e_n\| + c\Delta t_n^2 + \|(P_{n+1}-P_n)Q_n e_n\|$$
$$\le (1+C\Delta t)\|P_n e_n\| + C\Delta t_n^2 + \|P_{n+1}Q_n e_n\|.$$

From the boundedness of e_n, for $0 \le n \le N$, we obtain further that

$$\|P_{n+1}e_{n+1}\| \le (1+C\Delta t)^{n+1}\|P_0 e_0\| + C\Delta t^2 \sum_{l=0}^{n}(1+C\Delta t)^l$$
$$+C(\max_{0\le l\le n}\frac{1}{\Delta t_l}\|P_{l+1}Q_l\|)\Delta t \sum_{l=0}^{n}(1+C\Delta t)^l$$
$$\le C(\|P_0 e_0\| + \Delta t + \max_{0\le l\le n}\frac{1}{\Delta t_l}\|P_{l+1}Q_l\|),$$

which is the estimate (3.6).

The estimate (3.7) follows directly from (3.6). □

Remarks. Theorem 3.1 gives an error estimate for the recursive linearization procedure. The first two terms on the right hand side of (3.6) and (3.7) may be expected to be small. Further, observe that the last term in (3.6) and (3.7) is bounded. Actually, using Lemma 3.2

$$\|P_{n+1}Q_n\| = \|(P_{n+1}-P_n)Q_n\| \le \|P_{n+1}-P_n\| \le C\Delta t_n\ .$$

However, it is not clear whether in general

$$\frac{1}{\Delta t_n}\|P_{n+1}Q_n\| \to 0, \quad \text{as} \ \ \Delta t_n \to 0\ . \tag{3.8}$$

An interesting future project is to determine conditions under which the condition (3.8) or convergence of $q_N \to q(1)$ holds.

References

[1] Y. Chen, *Inverse scattering via Heisenberg's uncertainty principle,* Inverse Problem **13**(1997), 253-282.

[2] Y. Chen, *Inverse scattering via skin effect*, Inverse Problems **13** (1997), 647-667.

[3] Y. Chen and V. Rokhlin, *On the Riccati equations for the scattering matrices in two dimensions*, Inverse Problems.

[4] D. Isaacson, *Distinguishability of conductivities by electric current computed tomography*, IEEE Trans. Med. Imaging MI-5 (1986) 91-95.

[5] D. C. Dobson, *Estimates on resolution and stabilization for the linearized inverse conductivity problem*, Inverse Problems **8** (1992), 71-81.

[6] T. Kato, *Perturbation Theory for Linear Operators*, corrected 2nd ed., Springer-Verlag, Berlin, 1980.