Assignment 6

Problem 1

Suppose that $a(x), u(x)$ are sufficiently smooth. Denote by Δ_0 the standard central difference operator, namely, $\Delta_0 a(x) = a(x+h/2) - a(x-h/2)$. Show that
(a)

$$(a(x)u'(x))' = \frac{1}{h^2}\Delta_0(a(x)\Delta_0 u(x)) + O(h^2); \tag{1}$$

(b)

$$(a(x)u'(x))' = \frac{1}{2h^2}\{\, [a(x+h)+a(x)]\cdot[u(x+h)-u(x)] - [a(x)+a(x-h)]\cdot[u(x)-u(x-h)]\,\} + O(h^2). \tag{2}$$

Problem 2

Suppose that a, b, f are sufficiently smooth functions, and that a, b are also positive. Consider finite difference solution of the boundary value problem

$$\begin{aligned} -(a(x)u'(x))' + b(x)u(x) &= f(x), \quad x \in (0,1), &(3)\\ u'(0) &= 0, &(4)\\ u'(1) + \gamma u(1) &= \beta &(5) \end{aligned}$$

on the equispaced grid $x_j = jh$, $j = 0, 1, \ldots, m$, $h = 1/m$.
(a) Use (2) to write down second order finite difference approximation of the equation (3) at the interior grid points $x_j, j = 1, 2, \ldots, m-1$;
(b) Use the forward difference approximation (7.5) on pape 108 of the text to construct finite difference approximation of the left boundary condition (4) that is at least of second order (this is a finite difference equation at the first grid point x_0);
(c) Similarly use the backward difference approximation to construct finite difference approximation of the right boundary condition (5) that is at least of second order (this is a finite difference equation at the last grid point x_m);
(d) The linear system you established has $m+1$ equations for $m+1$ unknowns $u_j, j = 0, 1, \ldots, m$. It is tridiagonal except at the first and the last rows). Write a LU decomposition code for the solution in $O(m)$ steps, not in $O(m^3)$ step;
(d) Run your code for the given data

$$\begin{aligned} a(x) &= 1 + x\ln(2+x), &(6)\\ b(x) &= 1/(2+x), &(7)\\ f(x) &= (1+x^2)(1+\cos(15x)) &(8)\\ \gamma &= 1, \ \beta = 2, &(9) \end{aligned}$$

for $m = 20, 40, 80, 160, 320$ and check the rate of convergence with the L_2 norm $\|u\|_2 = (h\sum_{j=0}^{m} u_j^2)^{1/2}$;
(e) Plot the numerical solution in $[0,1]$ for $m = 160$.