Homework 3, Part II

Objective: Least-squares solution of under-determined linear system with the "physically relevant norms". Optimal design of narrow and wide bandwidth digital filters and high order quadrature formulae.

A filter f is required to be a smooth transition from 0 to 1, to be monotone ($f'(x) \geq 0$ for $x \in (a,b)$), and to have vanishing derivatives upto certain order at the two end points $a = -\pi,\ b = \pi$. If done correctly, the function

$$f_3(x) = \frac{1}{2\pi}(x+\pi) + \sum_{j=1}^{n} c_j \sin(jx) \tag{1}$$

will satisfy the requirements (except $f' \geq 0$) with suitably chosen coefficients c_j. In Homework 2, you have set up an m-by-n linear system for c_j, solved it with $m = n = 5$ and got a solution which gives rise to an f_3 that happens to be monotone.

There is no reason that we must choose $m = n$ and get a square linear system to which the solution is unique. On the contrary, we prefer $m < n$ so that the linear system is under-determined: there are more degrees of freedom than the number of equations. When this happens, the solution is not unique, and therefore there is freedom to choose a solution with desirable properties

Question 1. Let $D \in \mathbb{R}^{n\times n}$ be invertible. Describe with no more than 40 words (plus necessary formulae) a procedure which uses SVD to solve our problem $A \cdot c = b$ and simultaneously minimizes the 2-norm of $x = D \cdot c$. Hint: Consider the equation $A \cdot D^{-1} \cdot D \cdot c = b$. Note: for this problem to make practical sense, it is assumed that $m < n$ (what happens if $m = n$).

Question 2. Find out D if we want to solve our problem $A \cdot c = b$ and minimize the 2-norm of f_3', go to the end of this handout for details on the 2-norm of a constinuous function.

Question 3. For $m = 5,\ n = 8$, solve our problem $A \cdot c = b$ and minimize the 2-norm of $f_3^{(2m+2)}$. Show the condition number. Plot f_3 so constructed with 32 equispaced points in $[-\pi,\ \pi]$.

Question 4. For $m = 5,\ n = 31,\ \mu = 0.01$, solve our problem $A \cdot c = b$ and minimize $\|f_3'\|_2^2 + \mu\|f_3^{(4)}\|_2^2$. Show the condition number. Plot f_3 so constructed with 80 equispaced points in $[-\pi,\ \pi]$.

Appendix The 2-norm of a function f in $[a,b]$ is defined by the formula

$$\|f\|_2 = \left(\frac{1}{b-a}\int_a^b f^2(x)dx\right)^{\frac{1}{2}}; \tag{2}$$

therefore it can verified that a sine or cosine series

$$f(x) = \sum_{j=1}^{n} a_j \sin(jx), \quad \text{or} \quad f(x) = \frac{1}{2}a_0 + \sum_{j=1}^{n} a_j \cos(jx) \tag{3}$$

in $[a,b] = [-\pi,\ \pi]$ will have the 2-norm (up to a constant multiple)

$$\|f\|_2 = \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}}, \quad \text{or} \quad \|f\|_2 = \left(\sum_{j=0}^{n} |a_j|^2\right)^{\frac{1}{2}} \tag{4}$$