New York University
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G63.2701 *Handout #3*

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Assignment 2

Objective: Optimal design of digital filters and high order quadrature formulae.

1. Introduction

A digital filter (see Figure 1 at the end) is a function $f(x)$, $x \in [a,b]$ which provides a monotone, smooth transition from the state 0 to state 1; in other words, we require that for a positive ℓ

$$f(a) = 0,\ f(b) = 1,\ f'(x) \geq 0, \quad \text{(monotone in [a, b])} \tag{1}$$

$$f \in \mathbb{C}^{\ell}[a,b], \quad \text{(smooth in [a, b])} \tag{2}$$

$$\frac{d^j f}{dx^j}(a) = 0,\ \frac{d^j f}{dx^j}(b) = 0,\ j = 1,2,\ldots,\ell \quad \text{(smooth transition at } a \text{ and } b\text{)} \tag{3}$$

Let's from now on assume that

$$a = -\pi, \quad b = \pi. \tag{4}$$

A good filter requires a reasonably large ℓ, say, greater than 10, to ensure a smooth transition between 0 and 1. As examples, the functions

$$f_0(x) = \frac{x+\pi}{2\pi}, \quad x \in [-\pi,\pi] \tag{5}$$

$$f_1(x) = \frac{1}{2}\left[1 - \cos\left(\frac{x+\pi}{2}\right)\right] \quad x \in [-\pi,\pi] \tag{6}$$

are possible choices for a filter. But f_0 lacks the smoothness; it touches the floor 0 and the celling 1 like a stick, and Condition (3) is satisfied only for $\ell = 0$. This makes f_0 unsuitable as a filter. f_1 satisfies all Conditions (1)–(3) with $\ell = 1$, and therefore could be used as a filter (the oil industries use it for signal processing of their seismic data). But this filter delivers inferior sound quality if used in making CD's, partially because of the low ℓ value, and partially because the "physics" is not correctly built in.

A higher ℓ value is also needed for the so-called quadrature: a summation formula to approximate integrals. As we know, the most fundamental operation in scientific computing is the inner product. Nature does it in the form of integrals. So calculating an integral to a good precision is a bread-butter issue in scientific computing.

2. Description of the Problem

We will represent our filter, denoted by f_3, as a sum of two functions f_0 and p

$$f_3(x) = f_0(x) + p(x), \quad x \in [-\pi,\pi] \tag{7}$$

$$p(x) = \sum_{j=1}^{n} c_j \sin(jx), \quad x \in [-\pi,\pi] \tag{8}$$

where p is a periodic function in $[-\pi,\pi]$, and the Fourier coefficients c_j are to be determine so as to satisfy Condition (3). Note that Conditions (1), (2) are already fulfilled, except the requirement $f'(x) \geq 0$.

It is easy to verify that Condition (3) translates to

$$p'(\pi) = -\frac{1}{2\pi}, \tag{9}$$

$$p^{(2i-1)}(\pi) = 0, \quad i = 2,3,\ldots,m \tag{10}$$

where $m = \lfloor \ell/2 \rfloor$, since $p^{(2i)}(\pi)$ is automatically zero.

3. STATEMENT OF WORK

It is easy to see that the m equations in (9), (10) set up a system of linear equations for the n unknows $c = [c_1, c_2, \ldots, c_n]^T$; let's denote the system by $Ac = b$.

a. Write down expressions for $A(i,j)$, $b(i)$ for the m-by-n matrix A and the RHS b, for $1 \le i \le m$ and $1 \le j \le n$.

b. Write a matlab script for the solution of the linear system via SVD, under no assumption on m and n as which is greater than the other. Then solve for the unknown c in each of the three cases (i) $m = n = 5$ (ii) $m = n = 9$ and (iii) $m = 5$, $n = 9$.

c. Now with the Fourier coefficients c available, evaluate $f_3(x)$ and plot it with 50 equispaced x points in $[-\pi, \pi]$, for each of the three cases.

d. Repeat Steps (b) and (c) with the SVD replaced by a QR factorization.

e. Brief remarks (no longer than 120 words) on the results, as what is right and what may have gone wrong.

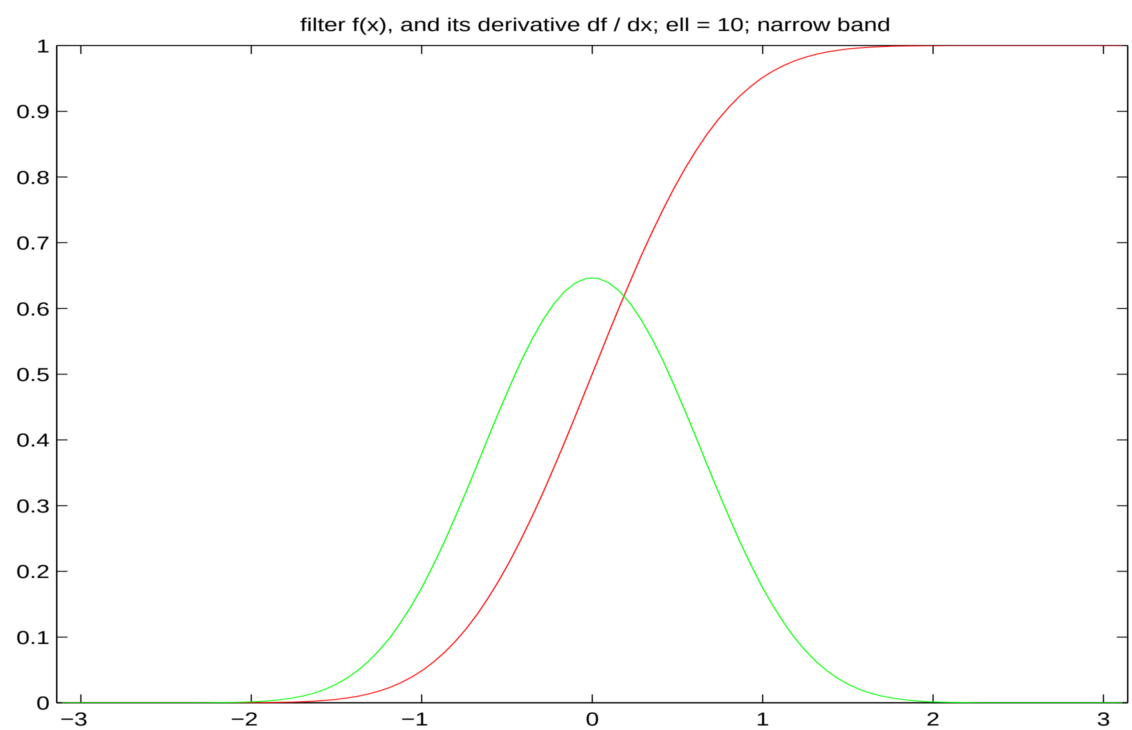

Figure 1: A typical plot of a filter f(x) and its first derivative