Real Variables Fall 2007.

Assignment 7. Due Oct 22.

Problem 1. $\{f_n\}$ is a sequence of integrable non-negative functions on a finite measure space and $f_n \to f$ almost everywhere. Moreover

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

so that equality holds in Fatou's lemma. Show that $\{f_n\}$ is uniformly integrable and

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0$$

Is it really necessary for $\mu(X)$ to be finite?

Problem 2.
(i.) $\{x_n\}$ is a sequence in a Hilbert space H such that for some $x \in H$

$$\lim_{n\to\infty} < y, x_n >=< y, x >$$

for all y. If in addition $\|x_n\| = \|x\| = 1$ for all n, show that $\|x_n - x\| \to 0$.
(ii.) If $\{f_n\}$ is sequence of functions in $L_2(X, \mathcal{B}, \mu)$ such that for some $f \in L_2(X, \mathcal{B}, \mu)$

$$\lim_{n\to\infty} \int_A f_n(x) d\mu = \int_A f(x) d\mu$$

for every $A \in \mathcal{B}$ and $\int_X |f_n|^2 d\mu = \int_X |f|^2 d\mu$ for all n, then show that

$$\int_X |f_n - f|^2 d\mu \to 0 \quad \text{as} \quad n \to \infty$$