3.3 Diffusions as Stocahstic Integrals

If $(\Omega\,, \mathcal{F}_t\,, P)$ is a probability space and $\beta(\cdot)$ is a d dimensional Brownian Motion relative to it, i.e. $\beta(t)$ is a diffusion with parameters $[0, I]$ relative to $(\Omega\,, \mathcal{F}_t\,, P)$, a stochastic integral $x(t)$ of the form

$$x(t) = \int_0^t b(s\,,\omega)ds + \int_0^t \sigma(s\,,\omega)d\beta(s) \tag{3.7}$$

is a diffusion with parameters $[b, \sigma\sigma^*]$. We want to show that the converse is true. Given a diffusion $x(t)$ on some $(\Omega\,, \mathcal{F}_t\,, P)$ corresponding to $[b, a]$ and given a progressively measurable σ such that $a = \sigma\sigma^*$, we want to show the existence of a Brownian motion $\beta(\cdot)$ on $(\Omega\,, \mathcal{F}_t\,, P)$ such that equation (3.7) holds. First let us remark that the converse as stated need not be true. For example if $(\Omega\,, \mathcal{F}_t\,, P)$ consists of a single point, P is the measure with mass 1 at that point, $x(t\,,\omega) \equiv 0$ definitely qualifies for a process corresponding to $[0, 0]$. No matter what Brownian Motion we take clearly

$$x(t) - x(0) = 0 = \int_0^t 0\, d\beta(s) \tag{3.8}$$

so the proposition must be trivially true. Except that the space is too small to support anything that is random, let alone a Brownian Motion. If we really need a Brownian Motion we have to borrow it. The way we borrow is to take a standard model of the Brownian Motion $(X\,, \mathcal{B}_t\,, Q)$ and take its product with $(\Omega\,, \mathcal{F}_t\,, P)$ as our new space. All the old previous processes are still there and replacing $\mathcal{F}_t$ with $\mathcal{F}_t \times \mathcal{B}_t$ does not destroy any of the previous martingale properties. But now we possess an extra Brownian Motion independent of everything. With the new borrowed Brownian Motion equation (3.8) is clearly true. One has to be careful with this sort of thing. We can only use such a totally arbitrary Brownian Motion when it does not matter what we use.

Let us describe the proof in different cases. First we assume that $a(s\,,\omega)$ is invertible almost surely and that σ is a square matrix with $\sigma\sigma^* = a$. Let us define

$$y(t) = x(t) - \int_0^t b(s\,,\omega)\, ds$$

and

$$z(t) = \int_0^t \sigma^{-1}(s\,,\omega)\, dy(s)$$

One can check that $z(\cdot)$ is well defined and has parameters $[0, I]$. This involves the calculation $\sigma^{-1} a \sigma = \sigma^{-1}\sigma\sigma^*\sigma^{-1*} = I$. $z(\cdot)$ is therefore Brownian Motion and

$$x(t) = x(0) + \int_0^t b(s\,,\omega)\, ds + \int_0^t \sigma(s\,,\omega)\, dz(s)$$

The next situation is when σ is a square matrix with $\sigma = \sqrt{a}$, with perhaps a singular somewhere. We now have to borrow a Brownian Motion and assume we have done it. Let $\pi(s,\omega)$ be the orthogonal projection on to the range of $a(s,\omega)$. The range of $\sigma(s,\omega)$ is the same as that of $a(\sigma,\omega)$ and we can construct the inverse $\tau(s,\omega)$ such that $\sigma(s,\omega)\tau(s,\omega) = \tau(s,\omega)\sigma(s,\omega) = \pi(s,\omega)$. We define $y(\cdot)$ as before. But we define $z(t)$ by

$$z(t) = \int_0^t \tau(s,\omega)\,dy(s) + \int_0^t [I - \pi(s,\omega)]d\beta(s)$$

where β is the borrowed d dimensional Brownian motion. It is only sparingly used. We note that $\sigma(s,\omega), \tau(s,\omega), \pi(s,\omega)$ and $[I - \pi(s,\omega)]$ are all symmetric.

$$\begin{aligned}
\tau(s,\omega)a(s,\omega)\tau(s,\omega) &+ [I - \pi(s,\omega)][I - \pi(s,\omega)] \\
&= \tau(s,\omega)\sigma(s,\omega)\sigma(s,\omega)\tau(s,\omega) + [I - \pi(s,\omega)][I - \pi(s,\omega)] \\
&= \pi(s,\omega)\pi(s,\omega) + [1 - \pi(s,\omega)][1 - \pi(s,\omega)] = I
\end{aligned}$$

So z is again Brownian Motion. We can now see that

$$x(t) = \int_0^t b(s,\omega)ds + \int_0^t \sigma(s,\omega)\,dz(s)$$

We need to show that

$$\int_0^t [\tau(s,\omega)dz(s) - I\,dy(s)] = 0$$

A mean square calculation leads to showing

$$(\tau(s,\omega)\sigma(s,\omega) - I)a(s,\omega)(\tau(s,\omega)\sigma(s,\omega) - I) + \tau(s,\omega)(I - \pi(s,\omega))\tau(s,\omega) = 0$$

which is identically true.

We can do the same thing when $\sigma(s,\omega)$ is given as an $n \times k$ matrix with $\sigma\sigma^* = a$ we now have to borrow a k dimensional Brownian Motion. We define a $k \times n$ matrix $\tau(s,\omega)$ by $\tau(s,\omega) = \sigma^*(s,\omega)a^{-1}(s,\omega)$, where $a^{-1}(s,\omega)$ is such that $a(s,\omega)a^{-1}(s,\omega) = a^{-1}(s,\omega)a(s,\omega) = \pi(s,\omega)$ with $\pi(s,\omega)$ as before. We denote by $\pi^*(s,\omega)$ the orthogonal projection on to the range of $\sigma^*(s,\omega)$. Now

$$z(t) = \int_0^t \tau(s,\omega)dy(s) + \int_0^t (I - \pi(s,\omega))d\beta(s)$$

The rest of the calculations go as follows.

$$\tau a\tau^* + (I - \pi) = \sigma^* a^{-1} a a^{-1}\sigma + (I - \pi) = \sigma^* a^{-1}\sigma + (I - \pi) = I$$

and

$$[I - \sigma\tau]a[I - \tau^*\sigma^*] + \tau[I - \pi]\tau^* = 0$$

3.4 Diffusions as Markov Processes

We will be intersted in defining Measures on the space $\Omega = C[[0,T];R^d]$ with the property that for some given $x_0 \in R^d$

$$P[x(0) = x_0] = 1 \tag{3.9}$$

and

$$Z_f(t\,,\omega) = f(x(t)) - f(x(0)) - \int_0^t (\mathcal{L}_s f)(x(s))ds \tag{3.10}$$

is a martingale with respect to $(\Omega\,,\mathcal{F}_t\,,P)$ for all smooth functions f, where

$$(\mathcal{L}_s f)(x) = \frac{1}{2}\sum_{i,j} a_{i,j}(s\,,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_i b_i(s\,,x)\frac{\partial f}{\partial x_i}(x) \tag{3.11}$$

The data are the starting point x_0 and the coefficients $a = \{a_{i,j}(s\,,x)\}$ and $b = \{b_i(s\,,x)\}$. We are seeking a solution P defined by these properties. In the earlier notation $a(s\,,\omega) = a(s\,,x(s\,,\omega))$ and $b(s\,,\omega) = b(s\,,x(s\,,\omega))$. Instead of starting at time 0 we could start at a different time s_0 from the point x_0 and then we would be seeking P, as a measure on the space $\Omega = C[[s_0,T];R^d]$ with analogous properties. We expect to show that under reasonable hypothese on the coefficients a and b, for each s_0, x_0, $P_{s_0\,,x_0}$ exists and is unique. The solutions $\{P_{s_0\,,x_0}\}$ will all be Markov Processes with continuous paths on R^d with transition probabilities

$$p(s\,,x\,,t\,,A) = P_{s,x}[x(t) \in A]$$

satisfying the Chapman-Kolmogorov equations

$$p(s\,,x\,,u\,,A) = \int p(s\,,x\,,t\,,dy)p(t\,,y\,,u\,,A)$$

for $s_0 \le s < t < u \le T$. Moreover

$$\begin{aligned} &P_{s,x}[x(t_1) \in A_1, \cdots, x(t_n) \in A_n] \\ &\qquad = \int_{A_1} \cdots \int_{A_n} p(s,x,t_1,dy_1) \cdots p(t_{n-1},y_{n-1},t_n,dy_n) \end{aligned} \tag{3.12}$$

Our goal is to find reasonably general conditions that guarantee the existence and uniqueness. We will then study properties of the solution P_{s_0,x_0} and how they are related to the properties of the coefficients.

3.5 Martingales and Conditioning.

Given $\mathcal{F}_s$ for some $s \in [0,T]$, we have the regular conditional probability distribution $Q_{s\,,\omega} = P|\mathcal{F}_t$ which has the following properties. For each ω, $Q_{s\,,\omega}$ is a probability measure on Ω and for each A, $Q_{s\,,\omega}(A)$ is $\mathcal{F}_s$ measurable. Moreover

$$Q_{s\,,\omega}[\omega' : x(t\,,\omega') = x(t\,,\omega) \text{ for } 0 \le t \le s] = 1$$

and

$$P(A) = \int Q_{s\,,\omega}(A) dP(\omega)$$

for all A.Such a Q exists and is essentially unique, i.e any two versions agree for almost all ω w.r.t P.

Lemma 3.6. *If $M(t)$ is a martingale relative to $(\Omega\,, \mathcal{F}_t\,, P)$ then for almost all ω and times $t \in [0,T]$, $M(t)$ is a martingale with respect to $(\Omega\,, \mathcal{F}_t\,, Q_{s,\omega})$.*

Proof. We need to check that if $A \in \mathcal{F}_{t_1}$, and $t_2 > t_1 \geq s$,

$$\int_A M(t_2) dQ_{s,\omega} = \int_A M(t_1) dQ_{s,\omega}$$

for almost all ω. Since both sides are $\mathcal{F}_s$ measurable it suffices to check that

$$\int_B \left[\int_A M(t_2) dQ_{s,\omega} \right] dP = \int_B \left[\int_A M(t_1) dQ_{s,\omega} \right] dP$$

From the properties of *rcpd* this reduces to

$$\int \left[\int_{A \cap B} M(t_2) dQ_{s,\omega} \right] dP = \int \left[\int_{A \cap B} M(t_1) dQ_{s,\omega} \right] dP$$

or

$$\int_{A \cap B} M(t_2) dP = \int_{A \cap B} M(t_1) dP$$

□

Since $A \cap B \in \mathcal{F}_{t_1}$ this follows from the martingale property of $M(t)$. Now all that is left is some technical stuff involving sets of measure 0. As of now the null set depends on $t_1\,, t_2$ and the set A. We take a countable set of rationals for $t_1\,, t_2$ and a countable set of $A's$ that generate the σ-field. One null set works for these. Any thing that works for these works for every thing by the usual bag of tricks. If we have a family $M_\alpha(t)$ of martingales indexed by α then the null set now may depend on α. If we can find a countable set of α's such that the corresponding set of $M_\alpha(t)$'s can approximate every $M_\alpha(t)$, we can get a single null set to work for all the martingales. The family $Z_f(t)$ indexed by smooth functions f is clearly such a family.

3.6 Conditioning and Stopping Times.

Lemma 3.7. *Let τ be a stopping time relative to the family $\mathcal{F}_t$ of σ-fields. We can apply the same reasoning to infer that for any martingale $M(t)$ the property, that it remains a martingale with respect to the r.c.p.d. $Q_{\tau\,,\omega}$ of P given $\mathcal{F}_\tau$ for times $t \geq \tau(\omega)$, is valid for almost all ω w.r.t. P.*

Proof. The proof again requires the verification for almost all ω of the relation

$$\int_A M(t_2) dQ_{\tau\,,\omega} = \int_A M(t_1) dQ_{\tau\,,\omega}$$

on the set $\{\omega : t_2 \geq t_1 \geq \tau(\omega)\}$. Given $B \in \mathcal{F}_\tau$ such that $B \subset \{\omega : \tau(\omega) \leq t_1\}$ we need to check

$$\int_B \left[\int_A M(t_2) dQ_{\tau\,,\omega}\right] dP = \int_B \left[\int_A M(t_1) dQ_{\tau\,,\omega}\right] dP$$

Since $B \subset \{\omega : \tau(\omega) \leq t_1\}$ and $B \in \mathcal{F}_\tau$ it follows from the definition of $\mathcal{F}_\tau$ that $B \in \mathcal{F}_{t_1}$ and it amounts to verifying

$$\int_{A\cap B} M(t_2) dP = \int_{A\cap B} M(t_1) dP$$

which follows from the facts $A \cap B \in M(t_1)$ and $M(t)$ is a P-martingale. One has to again do a song and dance regarding sets of measure zero. Ultimately, this reduces to the question: is $\mathcal{F}_\tau$ countably generated? The answer is yes, and in fact, it is not hard to prove that

$$\mathcal{F}_\tau = \sigma\{x(t \wedge \tau(\omega)) : t \geq 0\}$$

which is left as an exercise. $\square$

Let us suppose that we are given some coefficients $a(t\,,x)$ and $b(t\,,x)$. For each (s, x) we can define the class $\mathcal{M}_{s_0\,,x_0}$ as the set of solutions to the *Martingale Problem* for $[a, b]$, that start from the intial position x_0 at time x_o. A restatement of the result described earlier is that the r.c.p.d. $Q_{\tau\,,\omega}$ of $P|\mathcal{F}_\tau$ is again in the class $\mathcal{M}_{s\,,\tau}$. In particular if there is a unique solution $P_{s\,,\tau}$ to the martingale problem, then the r.c.p.d. $Q_{\tau\,,\omega} = P_{\tau,x(\tau)}$. This implies that once we have proved uniqueness, the solutions are all necessarily Markov and in fact strong Markov.

3.7 An easy example.

In R^d, let us take $a(t\,,x) = I$ and for $[I, b(t\,,x)]$ let us try to construct a solution to the martingale problem starting at (s_0, x_0). For simplicity let us assume that $b(t\,,x)$ is bounded uniformly. We can check that the expression

$$R_t(\omega) = \exp\left[\int_{s_0}^t < b(s\,,x(s)\,,dx(s) > -\frac{1}{2}\int_{s_0}^t \|b(s\,,x(s))\|^2 ds\right]$$

is a martingale with repect to $(\Omega\,,\mathcal{F}_t^{s_0}\,,Q_{s_0,x_0})$, where Q_{s_0,x_0} is the d-dimensional Brownian motion starting from x_0 at time 0. The same is true of

$$R_{\theta\,,t}(\omega) = \exp\left[\int_{s_0}^t < \theta + b(s\,,x(s))\,,dx(s) > -\frac{1}{2}\int_{s_0}^t \|\theta + b(s\,,x(s))\|^2 ds\right]$$

for every $\theta \in R^d$. We can write

$$R_{\theta\,,t}(\omega) = R_t(\omega) Z_{\theta\,,t}(\omega)$$

where

$$\begin{aligned} &Z_{\theta\,,t}(\omega) \\ &\quad = \exp\left[\int_{s_0}^t < \theta\,, dx(s) > - \int_{s_0}^t < \theta\,, b(s\,, x(s)) > - \frac{1}{2}\int_{s_0}^t \|b(s\,, x(s))\|^2 ds\right] \end{aligned}$$

We can define a measure P_{s_0,x_0} such that for $t \geq s_0$

$$\frac{dP_{s_0,x_0}}{dQ_{s_0,x_0}}\Big|_{\mathcal{F}_t^{s_0}} = R_t(\omega)$$

Then clearly $P_{s_0\,,x_0}$ is a solution. Conversely if P is any solution, Q defined by

$$dQ = [R_t(\omega)]^{-1} dP \quad \text{on} \quad \mathcal{F}_t^{s_0}$$

is a solution for $[I\,, 0]$ and is therefore the unique Brownian motion Q_{s_0,x_0}. Therefore $P = P_{s_0\,,x_0}$ defined above. So in this case we do have existence and uniqueness.

A second alternative is to try to solve the equation

$$y(t) = x(t) + \int_{s_0}^t b(s\,, y(s)) ds$$

for $t \geq s_0$ and for Brownian paths $x(t)$ that start from x_0 at time s_0. If we can prove existence and uniqueness, the solution will define a process which solves the martingale problem. It is defined on perhaps a larger space but it is easy enough to map the Wiener measure through $y(\cdot)$ and the transformed measure is a candidate for the solution of the martingale problem. Since we have uniquenes, this must coincide with the earlier construction. If b satisfies a Lipshitz condition in x this can be carried out essentially by Picard iteration.

A third alternative is to try to solve the PDE

$$\frac{\partial u}{\partial s} + \frac{1}{2}\Delta + \sum_i b_i(s\,, x)\frac{\partial u}{\partial x_i} = 0 \tag{3.13}$$

for $s \leq t$ with the boundary condition $u(t\,, x) = f(x)$. The fundamental solution $p(s\,, x\,, t\,, y)$ can be used as transition probabilities to construct a Markov Process which again is our old P. To see this, we verify that if u is any solution of equation (3.13) then $u(t\,, x(t))$ is a martingale with repect to any $P_{s_0\,,x_0}$ and therefore

$$u(s_0\,, x_0) = \int f(y) p(s_0\,, x_0\,, t\,, y) dy = E^{P_{s_0\,,x_0}}\left[f(x(t)\right]$$

Since this is true for any f the fundamental solution is the same as the transition probability of the alraedy constructed Markov Process.