General stochatsic differential equations and related PDE's.

Consider the SDE

$$dx(t) = \sigma(x(t)) \cdot d\beta(t) + b(x(t))\, dt \tag{1}$$

where $x(t) = \{x_1(t), x_2(t), \ldots, x_d(t)\} \in R^d$, $\beta(t) = \{\beta_1(t), \beta_2(t), \ldots, \beta_k(t)\}$ has k components and $\sigma(x) = \{\sigma_{i,r}(x)\}$ is a $d \times k$ matrix for each x. $b(x) = \{b_1(x), b_2(x), \ldots, b_d(x)\}$ has d components. this is to be viewed as the system

$$dx_i(t) = \sum_{r=1}^{k} \sigma_{i,r}(x(t))d\beta_r(t) + b_i(x(t))dt$$

with $x_i(0) = x_i$ for $1 \le i \le d$. Although usually $k = d$, it does not have to be. Associated with this system there is differential operator

$$\mathcal{L} = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

where

$$a_{i,j}(x) = \sum_{r=1}^{k} \sigma_{i,r}(x)\sigma_{j,r}(x)$$

or in matrix notation $a = \sigma\sigma^*$. Note that if $k < d$ the matrix a cannot be of full rank. However a is always positive semi-definite. If $\sigma(x)$ satisfies some regularity assumptions (Lipshitz condition) on the dependence on x the equations (1) will have a unique solution $x(t)$ that of course will depend on the starting point x. The process $x(t)$ satisfies these properties and can be characterized by suitable subsets of properties 1 through 8.

1. It is a Markov process with continuous paths, (no jumps).

2. The two moments of the transition probabilities $p(t, x, dy)$ satisfy

$$B_i(t, x) = \int (y_i - x_i)p(t, x, dy) = tb_i(x) + o(t)$$

$$A_{i,j}(t, x) = \int (y_i - x_i)(y_j - x_j)p(t, x, dy) = ta_{i,j}(x) + o(t)$$

and any higher moment

$$\delta(t, x) = \int |y - x|^\alpha p(t, x, dy) = o(t)$$

if $\alpha > 2$.

3. In fact

$$u(t, x) = \int f(y)p(t, x, dy) = f(x) + t(\mathcal{L}f)(x) + o(t)$$

where

$$(\mathcal{L}f)(x) = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x)$$

$\mathcal{L}$ is viewed as a differential operator acting on smooth function $f(x)$.

4. Note that while a is determined by σ the converse is not true. $a = \sigma\sigma^*$ has many solutions. These are different but equivalent models for the same process $x(t)$.

Remark. If a is nonsingular then $p(t, x, dy)$ will always have a density $p(t, x, y)dy$. Otherwise it may or may not. Try the example

$$dx_1(t) = d\beta_1(t), \; dx_2(t) = x_1(t)dt$$

with

$$a(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $b(x) = (x_1, 0)$. $x_1(t) = x_1 + \beta(t)$ and $x_2(t) = x_2 + t\,x_1 + \int_0^t \beta_1(s)ds$. The distribution of $x_1(t), x_2(t)$ is a nonsingular bivariate Gaussian.

5. The "Backward" Kolmogorov equations.

$$u(t, x) = \int f(y)p(t, x, dy)$$

stisfies

$$u_t = \mathcal{L}u$$

with $u(0, x) = f(x)$.

6. The "Forward" Kolmogorov equations. In the non-degenerate case assuming more regularity on a and b

$$v(t, y) = \int g(x)p(t, x, y)dx$$

satisfies

$$v_t = \mathcal{L}^* v$$

where $\mathcal{L}^a st$ is the adjoint of $\mathcal{L}$

$$\mathcal{L}^* g = \frac{1}{2}\sum_{i,j} \frac{\partial^2(a_{i,j}(x)g(x))}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial(b_i(x)g(x))}{\partial x_i}$$

defined by the relation

$$\int^{R^d} \mathcal{L}f \cdot g \; dx = \int_{R^d} f \cdot \mathcal{L}^* g \; dx$$

7. In fact $p(t, x, y)$ itself satsfies

$$p_t = \mathcal{L}p$$

as a function of t and x for fixed y and

$$p_t = \mathcal{L}^* p$$

as a function of t and y for fixed x. Note that y is the forward variable and x is the backward variable.

It is not hard to see (by differentiating with respect to t) that if u and v are solutions respectively of the backward and forward equations, then

$$\int u(T-t,x)v(t,x)dx$$

is independent of t in $0 \le t \le T$. In particular $E[f(x(t))]$ when $x(0)$ is random and distributed with density $g(x)dx$

$$E[g(x(T)] = \int f(x)v(T,x)dy = \int u(T,x)g(x)dx = E[u(T,x(0))]$$

Brownian motion corresponds to $a = I$, $b = 0$ or

$$\mathcal{L} = \mathcal{L}^* = \frac{1}{2}\Delta = \frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2}$$

8. Itô's formula holds.

$$\begin{aligned} du(t,x(t)) &= (\nabla u)(t,x(t)) \cdot dx(t) + u_t(t,x(t))dt + \frac{1}{2}\sum_{i,j} a_{i,j}(x(t))\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x(t))dt \\ &= (u_t + \mathcal{L}u)(t,x(t))dt + \nabla u \cdot \sigma(x(t)) \cdot d\beta(t) \end{aligned}$$

In particular

$$u(t,x(t)) = u(0,x) + \int_0^t [u_t + \mathcal{L}u](s,x(s))ds + M(t)$$

where $M(t)$ is a martingale.

9. Exit places. Dirichlet problem. Suppose G is a bounded open set in R^d with boundary ∂G and $x \in G$. Let

$$\tau = \inf\{t : x(t) \notin G\} = \inf\{t : x(t) \in \partial G\}$$

The expectation

$$u(x) = E[g(x(\tau))|x(0) = x]$$

is the unique solution of the Dirichlet Problem

$$\mathcal{L}u = 0, \ \ u = g \text{ on } \partial G$$

One can either solve the Dirichlet problem to find the exit distribution or the other way around.

One has to be sure that

$$P[\tau < \infty \,|x(0) = x] = 1 \text{ for all } x \in G$$

11. Possibility of Monte-Carlo simulation in order to solve the Dirichlet Problem.

12. Exit times and places. For $\lambda \geq 0$, the solution $u_\lambda(x)$ of

$$\mathcal{L}u = \lambda u, u(x) = g \text{ on } \partial G$$

is

$$u_\lambda(x) = E[e^{-\lambda\tau} g(x(\tau)) \,|x(0) = x]$$

Note that if $\lambda > 0$ we can have $\tau = \infty$ with positive probability. But if $\lambda = 0$ this cannot be allowed.

13. Regularity issues. Smooth solutions, weaker notions of solutions, distribution solutions etc.

14. In general, in the non-degenerate or elliptic case solutions are as smooth as the situation demands. For instance a solution of

$$(\mathcal{L}u)(x) = 0$$

will generally have two derivatives more than what the $\{a_{i,j}\}$ have.

Examples.

1. Geometric Brownian Motion.

$$dX(t) = \sigma X(t)d\beta(t) + \mu X(t)dt$$

Try

$$X(t) = X(0)\exp[\sigma\beta(t) + at]$$

Then

$$dX(t) = \sigma X(t)d\beta(t) + aX(t)dt + \frac{\sigma^2}{2}X(t)dt$$

$a = \mu - \frac{\sigma^2}{2}$ does it!. $X(t) = X(0)\exp[\sigma\beta(t) + (\mu - \frac{\sigma^2}{2})t]$. The solution of

$$u_t = \frac{x^2}{2}u_{xx} + \mu x u_x \;\; u(0,x) = f(x)$$

is therefore given by

$$u(t,x) = \int \frac{1}{\sqrt{2\pi t}} f(xe^{\sigma y + (\mu - \frac{\sigma^2}{2})t}) e^{-\frac{y^2}{2t}} dy$$

2. Stochastic volatility models:

$$dx(t) = \sigma(t)x(t)d\beta_1(t) + \mu x(t)dt$$
$$d\sigma(t) = f(\sigma(t))d\beta_2(t) + g(\sigma(t))dt$$

3. Two state volatality models. $\sigma = \{1, 2\}$. A Markov chain with rate matrix

$$\begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$$

$$\mathcal{L}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1 u_1 \\ \mathcal{L}_2 u_2 \end{pmatrix} + \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

4. Random Discount Models.

$$dx(t) = \sigma x(t)d\beta_1(t) - r(t)x(t)dt$$
$$dr(t) = f(r(t))d\beta_2(t) + g(r(t))dt$$