Dynamic programming;

Consider an asset allocation control problem:

$$dA(t) = A(t)[\sum_{i=1}^{n} \frac{u_i(t)}{x_i(t)} dx_i(t)]$$

where the asset A is allocated between n securities. The weights $u_i \geq 0$ and $\sum_i u_i = 1$. The securities which could include cash, are all Black-Scholes with the generator of $(x_1, \ldots, x_n)$ being

$$\frac{1}{2} \sum_{i,j} c_{i,j} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i x_i \mu_i \frac{\partial}{\partial x_i}$$

At the end of time T one wishes to maximize the utility $U(A) = |A|^p$ where $0 < p < 1$. What is the optimal allocation rule?

$$E[dA] = A \sum \mu_i u_i dt$$

$$E[(dA)^2] = \sum c_{i,j} u_i u_j dt$$

$$\mathcal{L}_u = \frac{1}{2} A^2 \sum c_{i,j} u_i u_j \frac{\partial^2}{\partial A^2} + A \sum \mu_i u_i \frac{\partial}{\partial A}$$

The equation to solve is

$$V_t + \sup_u \mathcal{L}_u V = 0, V(T, A) = A^p$$

Try $V(t, A) = v(t) A^p$.

$$v'(t) + \sup_u [\sum_i u_i \mu_i - \frac{1-p}{2} \sum c_{i,j} u_i u_j] v(t) = 0$$

the optimal u is a constant chosen according to the mean variance optimization scheme.

In general if we can solve

$$V_t + \sup_u \mathcal{L}_u V = 0, \quad V(T, x) = f(x)$$

then for any control $u(t, x)$ that depends only on the current state (policy) ,the corresponding process $x(t)$ will have the property

$$V_t + \mathcal{L}_{u(t,x)} V \leq 0, V(T, x) = f(x)$$

By maximum principle the actual solution

$$W_t + \mathcal{L}_{u(t,x)} W = 0, W(T, x) = f(x)$$

will have the property $W \leq V$. On the other hand for the optimizing u, V is the solution.