Maximum principle.

We will review our proof of the maximum principle from last week.

We consider a solution of

(1) $$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2}$$

on $[0,T] \times R$ with $u(0,x) = f(x)$. What do we mean by it?

1. For $t > 0$ and $x \in R$, u has one continuous derivative in t and two continuous derivatives in x and satisfies for $t > 0$ the equation (1).

2. $u(t,x)$ is continuous on $[0,T] \times R$ and $u(0,x) = f(x)$.

Theorem. Given $f(x)$ bounded and continuous, there exists a solution u which is bounded. A bounded solution is unique.

Proof: Existence:

$$u(t,x) = \int_R \frac{1}{\sqrt{2\pi t}} f(y) e^{-\frac{(x-y)^2}{2t}}\, dy$$

doest it.

Uniqueness depends on the maximum principle.

If u is a bounded solution such that $f(x) = u(0,x) \geq 0$ on R, then $u(t,x) \geq 0$ on $[0,T] \times R$. If we consider $v = u_1 - u_2$ then v will be a solution with $v(0,x) = 0$ and therefore $v \geq 0$ as well as $v \leq 0$ on $[0,T] \times R$, proving $V \equiv 0$ i.e. uniqueness.

The idea behind the maximum principle. Let u be a solution on $[0,T] \times [A,B]$ with with $u \geq 0$ on $\{0\} \times [A,B]$, $[0,T] \times \{A\}$ and $[0,T] \times \{B\}$. Suppose the minimum of $u(t,x)$ is attained at (t_0,x_0). If it is on the boundary $\{0\} \times [A,B]$, $[0,T] \times \{A\}$ and $[0,T] \times \{B\}$, then $u \geq 0$ throughout. Let us suppose it is either in the interior or on $\{T\} \times (A,B)$. In any case

$$u_t(t_0,x_0) \leq 0, u_x(t_0,x_0) = 0, u_{xx}(t_0,x_0) \geq 0$$

But $u_t = \frac{1}{2}u_{xx}$. This looks like a contradiction except that both u_t and u_{xx} may be zero at (t_0,x_0). We consider $v(t,x) = u(t,x)e^{-ct}$. Consider the point where v has a minimum. At that point $v_t(t_0,x_0) = u_t(t_0,x_0)e^{-ct} - cu(t_0,x_0)e^{-ct} \leq 0$. Moreover $v_{xx}(t_0,x_0) = u_{xx}(t_0,x_0)e^{-ct} \geq 0$. Since $u_t = \frac{1}{2}u_{xx}$, this yields $cu(t_0,x_0) \leq 0$. If $c > 0$, this implie that $u(t_0,x_0) \geq 0$ and we are done.

Another idea. Let us construct a solution $g_t = \frac{1}{2}g_{xx}$ that is non-negative, unbounded and grows rapidly when $x \to \pm\infty$. Example of one is

$$g(t,x) = \frac{1}{\sqrt{k-t}} e^{\frac{x^2}{2(k-t)}}$$

This is a solution. If $k > T$ this is a smooth solution on $[0,T] \times R$. If u is a bounded, in fact even unbounded so long as it does not grow too fast

$$u_\epsilon = u + \epsilon g$$

is a solution that is nonnegative on $\{0\} \times [-A, A]$, $[0, T] \times \{-A\}$ and $[0, T] \times \{A\}$ provided A is large enough. Therfore

$$u(t, x) + \epsilon g(t, x) \geq 0$$

for every $\epsilon > 0$. This will do.

The real idea behind the proof: Stochastics.

1. Itô's formula to $v(t, x) = u(T - t, x)$.

$$\begin{aligned} v(t, x(t)) &= v(0, x) + \int_0^t v_x(s, x(s)) dx(s) + \frac{1}{2} \int (v_t + \frac{1}{2} v_{xx})(s, x(s)) ds \\ &= \int_0^t v_x(s, x(s)) dx(s) \\ &= M(t) \end{aligned}$$

where $M(t)$ is a martingale. This requires

$$E[\int_0^T |v_x(t, x(t))|^2 dt | x(0) = x] < \infty$$

2. Stopping times: If we denote by

$$\tau_A = \inf\{t : |x(t)| = A\}$$

then

$$\begin{aligned} v(0, x) &= E_x[v(\tau \wedge T, x(T \wedge \tau))] \\ &= E_x[f(x(T)) : \tau > T] + E_x[u(T - \tau, -A) : \tau \leq T, x(\tau) = -A] \\ &\quad + E_x[u(T - \tau, -A) : \tau \leq T, x(\tau) = A] \end{aligned}$$

Note that

$$P[\tau < T] \leq 2 e^{-\frac{A^2}{2T}}$$

Therefore if

$$\lim_{A \to \infty} e^{-\frac{A^2}{4T}} \sup_{0 \leq t \leq T} [|u(t, A)| + |u(t, -A)|] = 0$$

then

$$v(0, x) = E_x[f(x(T)] = \int f(y) \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x-y)^2}{2T}} dy$$

proving uniqueness as well as the maximum principle.