1. Consider the PDE

$$u_t + \frac{1}{2}u_{xx} + b(x)u_x = 0, \quad u(1,x) = f(x)$$

with a smooth bounded b. Can you set up a difference scheme to solve it? How will you prove that your scheme works as the mesh size goes to 0?

2. Show that the solution of

$$u_t = \frac{1}{2}u_{xx} - xu_x, u(0,x) = f(x)$$

is given by

$$u(t,x) = \int f(y)p(t,x,y)dy$$

with

$$p(t,x,y) = \frac{1}{\sqrt{\pi(1-e^{-2t})}}e^{-\frac{(y-xe^{-t})^2}{(1-e^{-2t})}}$$

Calculate

$$\lim_{t\to\infty} u(t,x)$$

and show that it is independent of x. Show directly by differentiating the equation that for any function f with bounded derivative

$$\lim_{t\to\infty} \sup_x |u_x(t,x)| = 0$$

3. Consider the following problem of optimal control.

$$L_u = \frac{1}{2}\frac{d^2}{dx^2} + u\frac{d}{dx}$$

The goal is to minimize $E[(x(T))^2 + \frac{c}{2}\int_0^T u^2(s,x(s))ds]$ where

$$\frac{c}{2}\int_0^T u^2(s,x(s))ds$$

is the cost of the control. Here $c > 0$ is a constant. Consider the nonlinear equation

$$V_t + \frac{1}{2}V_{xx} - \frac{1}{2c}V_x^2 = 0, V(T,x) = x^2$$

Solve it. Try $V(t,x) = a(t) + \frac{b(t)}{2}x^2$. Show that the solution provides the minimum value and exhibit the control that achieves the minimum.

4. Consider the problem of controlling with a drift $b(t,x)$ with $|b(t,x)| \leq 1$, the expected exit time from the interval $[-1,1]$ of the process with generator

$$\frac{1}{2}\frac{d^2}{dx^2} + b(t,x)\frac{d}{dx}$$

aiming to minimize it. Show that the best control is the obvious one $b(t,x) = b(x) = sign\ x$, i.e. 1 if $x > 0$ and -1 if $x < 0$. [Hint: try solving $\frac{1}{2}u_{xx} + (sign\ x)u_x = -1$, $u(\pm 1) = 0$]. At 0 match so that $u_x(0) = 0$.