Test functions.

If we want to show that the Brownian motion in one dimension exits in a finite time from the interval $[1, 1]$, we know that the solution

$$\frac{1}{2}u_{xx} = -1, u(\pm 1) = 0$$

will give $E[\tau|X(0) = x]$. The solution is of course $(1 - x^2)$. In general it is not necessary to solve the equation explicitly. If we can find a function $u(x)$ such that $(\mathcal{L}u)(x) \geq c > 0$ in a region G, then for any starting point $x \in G$ the expected exit time from G is finite i.e.

$$E[\tau_G|X(0) = x] \leq \frac{2}{c} \sup_x |u(x)|$$

The proof uses Itô's formula to conclude that

$$u(x(t)) - \int_0^t (\mathcal{L}u)(x(s))ds$$

is a Martingale. Therefore if τ is a bounded stopping time such that $\tau \leq \tau_G$, then

$$E[u(x(\tau) - u(x) - c\tau|x(0) = x] \geq 0$$

In particular

$$E[\tau \wedge t|x(0) = x] \leq \frac{2}{c} \sup_x |u(x)|$$

Since this is true for every $t > 0$ by letting $t \to \infty$ we get our result.

Some times we need methods to conclude that $P[\tau < \infty] = 1$ while $E[\tau]$ may be infinite. If for some $c > 0$, we have a positive bounded function u on ∂G satisfying

$$(\mathcal{L}u)(x) - cu(x) \geq 0 \text{ for } x \in G$$

then

$$E[e^{-c\tau_G}|x(0) = x] \geq u(x)$$

In particular $P[\tau < \infty|x(0) = x] > 0$. To show that the probability is actually 1, we need to construct sub-solutions $u_c(x)$ such that $u_c(x) \to 1$ as $c \to 0$. The proof is again by Itô's formula.

$$d(e^{-ct}u(x(t))) = (\mathcal{L}u - cu)e^{-ct}dt + dM(t)$$

so that $e^{-ct}u(x(t))$ is a sub-martingale. In particular

$$E[e^{-c(\tau\wedge t)}u_c(x(\tau \wedge t))] \geq u_c(x)$$

Conversely if we have a super-solution with

$$(\mathcal{L}u)(x) - cu(x) \leq 0 \text{ for } x \in G$$

with $u(x) \to \infty$ as $x \to \partial G$, then$P[\tau < \infty | x(0) = x] = 0$. Follows from

$$E[e^{-c(\tau \wedge t)} u(x(\tau \wedge t))] \leq u(x)$$

or from the Martingale inequality

$$\sup_{0 \leq t < \tau} e^{-ct} u(x(t)) < \infty \quad \text{a.e.}$$

Allication: Non-explosion: If we can construct a function $u(x) > 0$ such that $u(x) \to \infty$ as $|x| \to \infty$ and

$$\sup_x \frac{(\mathcal{L}u)(x)}{u(x)} < \infty$$

then the process cannot explode.

Example: If $a(x) \leq C|x|^2$, $|b(x)| \leq C|x|$, then with $u(x) = 1 + |x|^2$,

$$\mathcal{L}u \leq Cu$$

Difference approximations to PDE

One way to numerically solve the heat equation

$$u_t + \frac{1}{2} u_{xx} = 0; u(T, x) = f(x)$$

is to approximate it by difference equations

$$\begin{aligned} &\frac{1}{\delta}[u((j+1)\delta, kh) - u(j\delta, kh)] \\ &\quad + \frac{1}{2h^2}[u((j+1)\delta, (k+1)h) + u((j+1)\delta, (k-1)h) - 2u((j+1)\delta, kh)] = 0 \end{aligned}$$

Time t marches in steps of size δ and the space x is made discrete with a spacing of h. Assuming $N\delta = T$, with $u(N\delta, kh) = f(kh)$, we iterate

$$\begin{aligned} &u(j\delta, kh) \\ &= u((j+1)\delta, kh) + \frac{\delta}{2h^2}[u((j+1)\delta, (k+1)h) + u((j+1)\delta, (k-1)h) - 2u((j+1)\delta, kh)] \\ &= \frac{\delta}{2h^2}[u((j+1)\delta, (k+1)h) + u((j+1)\delta, (k-1)h)] + (1 - \frac{\delta}{h^2})u((j+1)\delta, kh) \end{aligned}$$

We can let $\delta \to 0$, $h \to 0$ such that $\delta \leq h^2$. Then $u(j\delta, kh)$ will be an average of $u((j+1)\delta, (k \pm 1)h)$ and $u((j+1)\delta, kh)$. In particular if $\delta = h^2$

$$u_h(j\delta, kh) = \frac{1}{2}[u_h((j+1)\delta, (k+1)h) + u_h((j+1)\delta, (k-1)h)]$$

The convergence of $u_h(0,0)$ to the solution $u(0,0)$ of the heat equation given by

$$u(0,0) = \int f(y) \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} dy$$

is just the central limit theorem for the binomial distribution. Note that

$$u_h(0,0) = \sum_{r=0}^{N} \binom{N}{r} \frac{1}{2^N} f((2r-N)h)$$

where $h = \sqrt{\delta} = N^{-\frac{1}{2}}$.

We will give an alternate proof. Assume that f is smooth and the soution $u(t,x)$ of the heat equation has enough derivatives in t and x.

Then consider

$$\xi_n^h = u(n\delta, X_n^h)$$

where X_n^h is a Markov chain with transition probability

$$\pi_h(x, dy) = \frac{1}{2}\delta_{x+h}(dy) + \frac{1}{2}\delta_{x-h}(dy)$$

It is easily seen that (note $\delta = h^2$),

$$\begin{aligned}
E[u(n\delta, X_n^h)|X_{n-1}^h] &= \frac{1}{2}[u(n\delta, X_{n-1}^h + h) + u(n\delta, X_{n-1}^h - h)] \\
&= u(n\delta, X_{n-1}^h) + \frac{h^2}{2} u_{xx}(n\delta, X_{n-1}^h) + o(h^2) \\
&= u((n-1)\delta, X_{n-1}^h) + \delta u_t(n\delta, X_{n-1}^h) + \frac{h^2}{2} u_{xx}(n\delta, X_{n-1}^h) + o(\delta) \\
&= u((n-1)\delta, X_{n-1}^h) + o(\delta)
\end{aligned}$$

Therefore

$$E[f(X_N^h)|X_0^h = x] = E[u(T, x_N^h)|X_0^h = x] = u(0,x) + No(\delta) = u(0,x) + o(1)$$