Homework Set 6. Due March 29, 2004.

1. Consider the Brownian Motion $x(t) = x + \beta(t)$ starting from $x > 0$ at time 0. Show by two different methods that

$$P[\inf_{0\le s\le t} x(s) \ge 0] = \frac{\sqrt{2}}{\sqrt{\pi t}} \int_0^x e^{-\frac{y^2}{2t}} dy$$

a) Consider

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty [e^{-\frac{(x-y)^2 t}{2}} - e^{-\frac{(x+y)^2}{2t}}] f(y) dy$$

Check that

$$u_t = \frac{1}{2} u_{xx}$$

with $u(t,0) \equiv 0$ for $t > 0$ and $u(t,x) \to f(x)$ as $t \to 0$.

Check that for fixed T, by Itô's formula $u(T-t, x(t))$ is a martingale.

If $\tau = \inf\{t : x(t) \le 0\}$ is the first time 0 is reached, then verify

$$u(0,x) = E[u(T - \tau \wedge T, x(\tau \wedge T))] = E[f(x(\tau), \tau > T]$$

Take $f \equiv 1$.

b) Consider the function

$$u(t,x) = e^{-\lambda t - \sqrt{2\lambda}\ x}$$

Verify that

$$u_t + \frac{1}{2} u_{xx} = 0$$

and that $u(t, x(t))$ is a martingale. Deduce

$$e^{-\sqrt{2\lambda}\,x} = u(0,x) = E[e^{-\lambda\tau} | x(0) = x]$$

Complete the proof by verifying that the Laplace Transform

$$\int_0^\infty e^{-\lambda t} d\left[1 - \frac{\sqrt{2}}{\sqrt{\pi t}} \int_0^x e^{-\frac{y^2}{2t}} dy\right]$$
$$= \int_0^\infty e^{-\lambda t} dP[\tau \le t]$$
$$= e^{-\sqrt{2}\lambda\, x}$$