9. Second order Linear Partial differential equations.

If we model a stochastic process by

$$dx(t) = \sqrt{a(x(t))}d\beta(t) + b(x(t))dt \ ; \ x(0) = x$$

then we saw that $x(t)$ is a random process with continuous trajectories. If we have some path dependent payoff functional of the type

$$F(T, x(\cdot)) = \int_0^T e^{-\lambda s} V(x(s)) + e^{-\lambda T} f(x(T))$$

the payoff is random, and one is often interested in calculating the expected value of the payoff. More generally the model and the payoff could be explicitly time dependent

$$dx(t) = \sqrt{a(t, x(t))}d\beta(t) + b(t, x(t))dt \ ; \ x(s) = x$$

$$F(s, T, x(\cdot)) = \int_s^T \exp[-\int_s^t \lambda(\sigma, x(\sigma))d\sigma] V(t, x(t))dt + \exp[-\int_s^T \lambda(\sigma, x(\sigma))d\sigma] f(x(T))$$

and we wish to calculate

$$u(s, x) = E\Big[F(s, T, x(\cdot))|x(s) = x\Big]$$

as a function of (s, x). From the Markov property it is clear that for times $s_1 < s_2$,

$$u(s_1, x) =$$
$$E\Big[\int_{s_1}^{s_2} e^{-\int_{s_1}^t \lambda(\sigma, x(\sigma))d\sigma} V(t, x(t))dt + e^{-\int_{s_1}^{s_2} \lambda(\sigma, x(\sigma))d\sigma} u(s_2, x(s_2))|x(s_1) = x\Big]$$

We think of $s_1 = s$ and $s_2 = s + \epsilon$. Then

$$\begin{aligned} &u(s, x) - u(s + \epsilon, x) \\ &\simeq E\Big[\epsilon V(s, x(s)) + (1 - \epsilon\lambda(s, x(s)))u(s + \epsilon, x(s + \epsilon))|x(s) = x\Big] \\ &\simeq \epsilon[V(s, x) - \lambda(s, x)u(s, x)] + E\Big[u(s + \epsilon, x(s + \epsilon)) - u(s + \epsilon, x)|x(s) = x\Big] \end{aligned}$$

Note that for any smooth function u

$$
\begin{aligned}
&E\Big[u(x(s+\epsilon) - u(x(s))|x(s) = x\Big] \\
&\simeq E\Big[u_x(x)(x(s+\epsilon) - x(s)) + \frac{1}{2}u_{xx}(x)(x(s+\epsilon) - x(s))^2|x(s) = x\Big] \\
&\simeq \epsilon[b(s,x)u_x(x) + \frac{a(s,x)}{2}u_{xx}(x)]
\end{aligned}
$$

Therefore

$$u(s,x) - u(s+\epsilon,x) = \epsilon[V(s,x) - \lambda(s,x)u(s,x) + b(s,x)u_x(s,x) + \frac{a(s,x)}{2}u_{xx}(s,x)]$$

Or

$$(1)\ \ u_s(s,x)+b(s,x)u_x(s,x)+\frac{a(s,x)}{2}u_{xx}(s,x)-\lambda(s,x)u(s,x)+V(s,x) = 0\,;u(T,x) = f(x)$$

Equations of the above type are called backward parabolic equations. The **MAXIMUM PRINCIPLE** states that if

1. u, u_x, u_{xx} are bounded and continuous and u satisfies equation (1) in $[0,T] \times R$,
2. b and a are bounded and continuous and $a \geq 0$,
3. $V(s,x) \geq 0$ for all (s,x) and $f(x) \geq 0$ for all x,

then $u(s,x) \geq 0$ for all $(s,x) \in [0,T] \times R$.

Proof of the maximum principle. The basic idea is to show that solutions of (1) cannot achieve their minimum except when $s = T$. Since $u(T,x) = f(x) \geq 0$ this will imply that $\min u(s,x) \geq 0$ and we are done. If the minimum is attained at some (s_0,x_0) with $s_0 < T$, then at that point $u_s(s_0,x_0) \geq 0$ and $u_x(s_0,x_0) = 0$. Moreover $u_{xx}(s_0,x_0) \geq 0$. If only the inequality $\lambda(s_0,x_0) \geq 1$ was true, we would be done. All the terms in the equation are nonnegative and they add up to 0. Since $\lambda(s_0,x_0) \geq 1$, we must have $u(s_0,x_0) \geq 0$. It is easy to achieve $\lambda(s,x) \geq 1$ with out changing the problem. Instead of u we consider the function $v(s,x) = u(s,x)e^{C(s-T)}$ with a constant C to be chosen later. Then v will satisfy

$$
\begin{aligned}
&v_s(s,x) + b(s,x)v_x(s,x) + \frac{a(s,x)}{2}v_{xx}(s,x) - [C + \lambda(s,x)]v(s,x) + V(s,x)e^{C(s-T)} = 0 \\
&v(T,x) = f(x)
\end{aligned}
$$

If we pick C large enough, the new λ which is $C+\lambda$, can be assumed to be larger than 1. The new V which is $V(s,x)e^{C(s-T)}$ is nonnegative since the old one was. Now we will be able to conclude that at the new (s_0,x_0) where v achieves the minimum we must have $v(s_0,x_0) \geq 0$ and therefore $v(s,x) \geq 0$ for all (s,x). This will imply that $u(s,x) \geq 0$ as well. This proof still needs to be fixed. Since x varies over an unbounded set the infimum may not be attained. We replace $u(t,x)$ by a new $v(t,x)$ where

$$v(t,x) = u(t,x)e^{-\epsilon h(x)+C(s-T)}$$

Think of $h(x)$ as $\sqrt{1+x^2}$. The function v vanishes as $|x| \to \infty$ and if it is not nonengative must now necessarily achieve its negative minimum at some point (s_0,x_0) with $s_0 < T$. At this point $u(s_0,x_0) < 0$, $v(s_0,x_0) < 0$, $v_s(s_0,x_0) \geq 0$, $v_x(s_0,x_0) = 0$ and $v_{xx}(s_0,x_0) \geq 0$. Since

$$u(t,x) = v(t,x)e^{\epsilon h(x)-C(s-T)}$$

$$\begin{aligned}
u_s(s_0,x_0) &= [v_s(s_0,x_0) - Cv(s_0,x_0)]e^{\epsilon h(x_0)-C(s_0-T)} \geq -Cu(s_0,x_0) \\
u_x(s_0,x_0) &= [v_x(s_0,x_0) + \epsilon h'(x)v(s_0,x_0)]e^{\epsilon h(x_0)-C(s_0-T)} = \epsilon h'(x)u(s_0,x_0) \\
&\geq B\epsilon u(s_0,x_0) \\
u_{xx}(s_0,x_0) &= [v_{xx}(s_0,x_0) + 2\epsilon h'(x_0)v_x(s_0,x_0) + \epsilon h''(x_0)v(s_0,x_0) + \epsilon^2[h'(x_0)]^2 v(s_0,x_0)] \\
&\qquad \times e^{\epsilon h(x_0)-C(s_0-T)} \\
&\geq B[\epsilon + \epsilon^2]u(s_0,x_0)
\end{aligned}$$

where B is an upper bound on $|h'(x)|$ and $|h''(x)|$. Finally substituting in equation (1)

$$\begin{aligned}
0 &= u_s(s,x) + b(s,x)u_x(s,x) + \frac{a(s,x)}{2}u_{xx}(s,x) - \lambda(s,x)u(s,x) + V(s,x) \\
&\geq -[C + \lambda(s_0,x_0)]u(s_0,x_0) + K[B\epsilon + \frac{B}{2}\epsilon^2]u(s_0,x_0)
\end{aligned}$$

where K is an upper bound on $|b(s,x)|$ and $a(s,x)$. If $C + \lambda(s,x) \geq 1$, for ϵ small enough $K[B\epsilon + \frac{B}{2}\epsilon^2] \leq \frac{1}{2}$. Proving that $u(s_0,x_0) \geq 0$. This implis that $v(s_0,x_0) \geq 0$ which in turn implies $v(s,x) \geq 0$ for all (s,x) and $u(s,x) \geq 0$ for all (s,x)

The maximum principle in particular implies uniqueness. If for given a, b, λ, V and f we have two solutions u and v, the difference $w = u - v$ will be a solution for the same a, b and λ but with $V \equiv f \equiv 0$. It now follows from the maximum principle that w and $-w$ are nonnegative. Hence $w \equiv 0$ or $u \equiv v$.

Actually one can use the theory of stochastic differential equations to provide a more direct proof of the maximum principle. Let us suppose that u is a bounded continuous function, on $[0,T] \times R$ with enough derivatives (two in x and one in s), that satisfies

$$u_s + b(s,x)u_x + \frac{a(s,x)}{2}u_{xx} - \lambda(s,x)u + V(s,x) = 0$$

Then the function

$$F(t) = u(t,x(t))e^{-\int_0^t \lambda(s,x(s))ds}$$

where $x(t)$ is a solution of

$$dx(t) = \sqrt{a(t,x(t))}d\beta(t) + b(t,x(t))dt$$

will satisfy

$$\begin{aligned} dF(t) &= [-\lambda(t,x(t))u(t,x(t))dt + u_x dx(t) + u_t dt + \frac{1}{2}u_{xx}(dx(t))^2]e^{-\int_0^t \lambda(s,x(s))ds} \\ &= [-\lambda u dt + b u_x dt + \sqrt{a}d\beta(t) + u_t dt + \frac{a}{2}u_{xx}dt]e^{-\int_0^t \lambda(s,x(s))ds} \\ &= -V(t,x(t))e^{-\int_0^t \lambda(s,x(s))ds}dt + e^{-\int_0^t \lambda(s,x(s))ds}\sqrt{a(t,x(t))}d\beta(t) \end{aligned}$$

Or

$$F(T) - F(0) + \int_0^T V(t,x(t))e^{-\int_0^t \lambda(s,x(s))ds}dt + \int_0^T e(s)d\beta(s)$$

for some e. In particular this has mean zero. $F(0)$ is a constant and equals $u(0,x)$. Hence

$$\begin{aligned} u(0,x) &= E\Big[F(T) + \int_0^T V(t,x(t))e^{-\int_0^t \lambda(s,x(s))ds}dt | x(0) = x\Big] \\ &= E\Big[f(x(T))e^{-\int_0^T \lambda(s,x(s))ds} + \int_0^T V(t,x(t))e^{-\int_0^t \lambda(s,x(s))ds}dt | x(0) = x\Big] \end{aligned}$$

The above relationship between the solution u of a PDE and expectations of certain path dependent functions of solutions $x(\cdot)$ of an SDE is a crucial link between the two. We provided a proof based essentially on Itô's formula that computed

$$dF(t) = h(t)dt + H(t)d\beta(t)$$

and because any integral $\int_{t_1}^{t_2} H(s)d\beta(s)$ had expectation 0, we concluded that

$$E\Big[F(t_2) - F(t_1) - \int_{t_1}^{t_2} h(s)ds \big| x(t_1) = x\Big] = 0$$

for any $t_1 < t_2$ and x. By the Markov property this implies that

$$E\Big[F(t_2) - F(t_1) - \int_{t_1}^{t_2} h(s)ds \big| x(t) = x\Big] = 0$$

so long as $t \le t_1 < t_2$. We can avoid the explicit use of Itô's formula if we want. Consider the quantity

$$k(t) = E\Big[u(t,x(t))e^{-\int_0^t \lambda(s,x(s))ds} + \int_0^t V(s,x(s))e^{-\int_0^s \lambda(\sigma,x(\sigma))d\sigma} \big| x(0) = x\Big]$$

We will show that $k(t)$ is a constant as a function of t. In particular

$$\begin{aligned}u(0,x) &= k(0) = k(T)\\ &= E\Big[f(x(T))e^{-\int_0^T \lambda(s,x(s))ds} + \int_0^T V(s,x(s))e^{-\int_0^s \lambda(\sigma,x(\sigma))d\sigma}ds\big|x(0)=x\Big]\end{aligned}$$

It is clearly sufficient to calculate $k'(t)$ and show that it is identically 0. We will estimate $k(t+h)-k(t)$ and see why this is $o(h)$ due to cancellations and not $O(h)$. It is enough to show

$$\begin{aligned}&E\Big[(u(t+h,x(t+h))-u(t,x(t)))e^{-\int_0^t \lambda(s,x(s))ds}\big|x(0)=x\Big]\\ &\quad+E\Big[u(t,x(t))e^{-\int_0^t \lambda(s,x(s))ds}(e^{-\int_t^{t+h} \lambda(s,x(s))ds}-1)\big|x(0)=x\Big]\\ &\quad+E\Big[\int_t^{t+h} V(s,x(s))e^{-\int_0^s \lambda(\sigma,x(\sigma))d\sigma}ds\big|x(0)=x\Big]=o(h)\end{aligned}$$

We first condition the path $x(s)$ upto time t. This gives a common factor of $e^{-\int_0^t \lambda(s,x(s))ds}$ that can be pulled out leaving for us to show that

$$\begin{aligned}&E\Big[(u(t+h,x(t+h))-u(t,x(t)))\big|x(t)=x\Big]\\ &\quad+E\Big[u(t,x(t))(e^{-\int_t^{t+h} \lambda(s,x(s))ds}-1)\big|x(t)=x\Big]\\ &\quad+E\Big[\int_t^{t+h} V(s,x(s))e^{-\int_t^s \lambda(\sigma,x(\sigma))d\sigma}ds\big|x(t)=x\Big]=o(h)\end{aligned}$$

The second and third term are easy. They yield

$$[-u(t,x(t))\lambda(t,x(t))+V(t,x(t))]h+o(h)$$

The first term has to be expanded by Taylor's formula

$$\begin{aligned}u(t+h,x(t+h))-u(t,x(t)) &= u_t(t,x)h+u_x(t,x)E[(x(t+h)-x(t))|x(t)=x]\\ &\quad+\frac{1}{2}u_{xx}(t,x)E[(x(t+h)-x(t))^2|x(t)=x]\\ &= h[u_t(t,x))+b(t,x)u_x(t,x)+\frac{1}{2}a(t,x)u_{xx}(t,x)]+o(h)\end{aligned}$$

Since u satisfies the equation

$$u_t(t,x)+b(t,x)u_x(t,x)+\frac{1}{2}a(t,x)u_{xx}(t,x)-\lambda(t,x)u(t,x)+V(t,x)=0$$

all the $O(h)$ terms cancel out to give $k'(t)\equiv 0$.

The representation of the solution as an expectation shows that u is nonnegative if f and V are. It also proves uniqueness. In particular if we pick $V \equiv \lambda \equiv 0$, then the solution of

$$dx(t) = \sqrt{a(t, x(t))}d\beta(t) + b(t, x(t))dt : x(s) = x$$

and

$$u_s(s, x) + b(s, x)u_x(s, x) + \frac{a(s, x)}{2}u_{xx}(s, x) = 0 \; ; u(T, x) = f(x) \tag{2}$$

are related by

$$u(s, x) = E\big[f(x(T))|x(s) = x\big]$$

If we denote by $p(s, x, t, y)dy$ the transition probability density $P[x(t) \in dy|x(s) = x]$ then

$$u(s, x) = \int f(y)p(s, x, T, y)dy \tag{3}$$

One can use results from PDE that tell us that the equation (2) has a nice solution and in fact the solution is given by a formula of the type (3). The densities $p(s, x, t, y)$ satisfy

$$p(s, x, t, y) \geq 0 \; ; \int p(s, x, t, y)dy = 1$$

and for $s < \sigma < t$

$$\int p(s, x, \sigma, z)p(\sigma, z, t, y)dz = p(s, x, t, y)$$

One can then construct directly a Markov process with transition probabilities $\{p(s, x, t, y)\}$ This will be statistically the same as the solution of the stocahstic differential equation.

Examples:

1. Consider the SDE

$$dx(t) = dt + d\beta(t) \; ; x(s) = x$$

with a solution $x(t) = x + \beta(t) - \beta(s) + t - s$. The probability density of $x(t)$ is Gaussian with mean $x + t - s$ and variance $t - s$ and is given by

$$p(s, x, t, y) = \frac{1}{\sqrt{2\pi(t - s)}}e^{-\frac{(y-x-t+s)^2}{2(t-s)}}$$

One can check that for fixed t and y, p satisfies

$$p_s + \frac{1}{2}p_{xx} + p_x = 0$$

2*. For the linear stochastic differential equation

$$dx(t) = bx(t)dt + \sigma x(t)d\beta(t)$$

calculate explicitly the probability density $p(s,x,t,y) = p(t-s,x,y)$ by solving the stochastic differential equation. For any $\alpha \in R$ calculate the integral

$$u_\alpha(t,x) = \int_0^\infty y^\alpha p(t,x,y)dy$$

Verify that u_α satisfies the equation

$$u_t = bxu_x + \frac{\sigma^2 x^2}{2} u_{xx}$$

3*. Solve explicitly the stochastic differential equation

$$dx(t) = -cx(t)dt + d\beta(t); x(0) = x$$

Find $p(t,x,y)$. Show that for each fixed y, $p(t,x,y)$ satisfies the equation

$$p_t = -cxp_x + \frac{1}{2}p_{xx}$$

The functions $p(s,x,t,y)$ or in the time homogeneous case, $p(t-s,x,y)$ satisfiy the equation

$$p_s(s,x) + b(s,x)p_x(s,x) + \frac{a(s,x)}{2}p_{xx}(s,x) = 0$$

In particular any integral $\int p(s,x,t,y)f(y)dy$ will satisfy

$$u_s(s,x) + b(s,x)u_x + \frac{a(s,x)}{2}u_{xx}(s,x) = 0$$

with

$$\int f(y)p(s,x,t,y)dy \to f(x)$$

as $s \to t$. In PDE they are called fundamental solutions and yield directly the transition probabilities.