8. Stocahstic Differential Equations as limits of Markov Chains

Instead of a random walk which has increments or steps whose distributions are independent of their current position, we can have Markov Chains moving in R, that take small steps, but the distribution of the steps depend on the current position of the Markov Chain. We think of a small parameter $h > 0$ as the unit of time and the Markov Chain from the position X_n^h at time nh moves to its next position X_{n+1}^h with a step of or increment of $Y_{n,n+1}^h = X_{n+1}^h - X_n^h$. We anticipate that in the limit as $h \to 0$ only the mean and variance of the increment $Y_{n,n+1}^h$ will matter. Assuming the transition probabilities to be stationary in time, we denote by

$$\begin{aligned} b^h(x) &= E[Y_{n,n+1}^h | X_n^h = x] \\ a^h(x) &= E[(Y_{n,n+1}^h)^2 | X_n^h = x] \\ \Delta^h(x) &= E[|Y_{n,n+1}^h|^3 | X_n^h = x] \end{aligned}$$

We saw earlier that if $b^h(x) = o(h), a^h(x) = h + o(h)$ and $\Delta^h(x) = o(h)$, then the distribution of X_n^h coverges to a Normal distribution with mean $X_0 = x$ and variance t, provided $nh \to t$. One can improve this to the convergence of X_n^h to the Brownian Motion $x + \beta(t)$, in the sense that the joint distributions of $\{X_{n_i}^h : 1 \le i \le k\}$ converges to the joint distributions of $\{x + \beta(t_i) : 1 \le i \le k\}$ provided $n_i h \to t_i$ for $i = 1, \cdots, k$. We will now assume that

$$\begin{aligned} b^h(x) &= hb(x) + o(h) \\ a^h(x) &= ha(x) + o(h) \\ \Delta^h(x) &= o(h) \end{aligned}$$

Although $a^h(x)$ is only the second moment and not the variance, the difference which is the square of the mean is $(b^h(x))^2$ and is $O(h^2) = o(h)$ and can be ignored. One way to model such a situation (by no means unique) is to assume

$$X_{n+1}^h = X_n^h + hb(x) + \sqrt{a(x)}\sqrt{h}\xi_n$$

where $\{\xi_n\}$ i.i.d. standard normals. Or one can replace $\sqrt{h}\xi_n$ by $\beta((n+1)h) - \beta(nh)$ to get

$$X_{n+1}^h = X_n^h + hb(X_n^h) + \sqrt{a(X_n^h)}\beta((n+1)h) - \beta(nh))$$

We can take a formal limit here to arrive at

$$dX(t) = b(X(t))dt + \sqrt{a(X(t))}d\beta(t)$$

This equation cannot be treated as a standard ODE. $\beta(t)$ as we saw is not of bounded variation and even in the integrated form

$$X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \sqrt{a(X(s))}d\beta(s) \tag{1}$$

does not make sense at the first glance.

There is a theory developed by K.Itô that treats this. The main ideas are the following steps the details of whch we will not go into.

Step 1. Approximate inegrals of the form $\int_0^t F(s)d\beta(s)$ by

$$\mathcal{I}_n = \sum_{j=0}^{n-1} F(t_j)[\beta(t_{j+1}) - \beta(t_j)] : 0 = t_0 < t_1 < \cdots < t_n = t$$

sticking the increments always in the future. If $F(s)$ only depends on the past history upto time t then $F(t_j)$ is independent of $\beta(t_{j+1}) - \beta_(t_j)$ and a simple calculation yields

$$E[\mathcal{I}_n] = 0$$
$$E[\mathcal{I}_n^2] = \sum_{j=0}^{n-1} E[F^2(t_j)(t_{j+1} - t_j)]$$

suggesting a definition of

$$\mathcal{I} = \int_0^t F(s)d\beta(s)$$

for random functions $F(s)$ that depend only on past history such that

$$E[\mathcal{I}] = 0$$
$$E[\mathcal{I}^2] = E[\int_0^t F^2(s)ds]$$

Step 2. Define iteratively

$$X_{n+1}(t) = x + \int_0^t b(X_n(s))ds + \int_0^t \sqrt{a(X_n(s))}d\beta(s)$$

Step 3. Using the above iteration, similar to Picard iteration for ODE, prove that $X_n(\cdot)$ has a limit $X(\cdot)$, that satisfies the equation (1). Prove uniqueness. One makes the asummption that $b(x)$ and $\sqrt{a(x)}$ satisfy the Lipshitz condition

$$|b(x) - b(y)| \leq A|x - y|$$
$$|\sqrt{a(x)} - \sqrt{a(y)}| \leq A|x - y|$$

Step 4. Develop a calculus. (Itô Calculus). If we expand

$$\begin{aligned} f(\beta(t)) - f(\beta(0)) &= \sum [f(\beta((j+1)h) - f(\beta(jh))] \\ &= \sum_j f'(\beta(jh)[\beta(j+1)h) - \beta(jh)] \\ &\quad + \sum_j \frac{1}{2} f''(\beta(jh))[\beta(j+1)h) - \beta(jh)]^2 \\ &\quad + \sum_j O(|\beta(j+1)h) - \beta(jh)|^3) \\ &= \int_0^t f'(\beta(s))d\beta(s) + \frac{1}{2}\int_0^t f''(\beta(s))ds \end{aligned}$$

We have used the properties that refining an interval $[0, t]$ into finer and finer partitions leads to

$$\sum [\beta(t_{j+1} - \beta(t_j)]^2 \to t$$

and

$$\sum |\beta(t_{j+1} - \beta(t_j)|^3 = nO(n^{-\frac{3}{2}}) \to 0$$

Formally $[d\beta(t)]^2 = dt$ and $[d\beta(t)]^k = 0$ for $k \geq 3$. In Taylor expansion we always keep two terms. Any mixed term $d\beta dt$ is equal to 0. With this rule one can start with

$$dX(t) = b(X(t))dt + \sqrt{a(X(t))}d\beta(t)$$

and get

$$[dX(t)]^2 = a(X(t))dt$$

or

$$\begin{aligned} du(t, X(t)) &= u_t(t, X(t))dt + u_x(t, X(t))dX(t) + \frac{1}{2}a(X(t))u_{xx}(t, X(t))dt \\ &= u_t(t, X(t))dt + u_x(t, X(t))[\sqrt{a(X(t))}d\beta(t) + b(X(t))dt] \\ &\quad + \frac{1}{2}a(X(t))u_{xx}(t, X(t))dt \end{aligned}$$

This is to be interpreted as the identity

$$\begin{aligned} u(t, X(t)) - u(0, x) &= \int_0^t u_x(s, X(s))\sqrt{a(X(s))}d\beta(s) \\ &\quad + \int_0^t g(s, X(s))ds \end{aligned}$$

with $g(t, x) = u_t(t, x) + b(x)u_x(t, x) + \frac{1}{2}a(x)u_{xx}(t, x)$.

Examples.

1. Let $f(x) = x^2$. Then

$$\beta(t)^2 - \beta(0)^2 = 2\int_0^t \beta(s) d\beta(s) + t$$

This can be directly verified by approximation and using the relation

$$\sum_j [\beta(t_{j+1}) - \beta(t_j)]^2 \to t$$

2*. Show that the solution of $dX(t) = X(t)d\beta(t)$ is $X(t) = X(0)\exp[\beta(t) - \frac{t}{2}]$ and not $X(0)\exp[\beta(t)]$.

3*. If $u(t,x)$ satisfies the PDE

$$u_t(t,x) + b(x)u_x(t,x) + \frac{a(x)}{2}u_{xx}(t,x) \equiv 0 \quad \text{for} \quad 0 \leq s \leq T \quad \text{and} \quad u(T,x) = f(x)$$

and $X(t)$ satisies

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sqrt{a(X(s))}d\beta(s)$$

then use Itô calculus and the fact that $E[\int_0^T F(s)d\beta(s)] = 0$, to show that

$$u(0,x) = E[f(X(T))]$$

Remark: Technically one needs to know that

$$E[\int_0^T [F(s)]^2 ds] < \infty$$

to define the integral $\int_0^T F(s)d\beta(s)$ and show that it has mean 0 and its variance is equal to $E[\int_0^T [F(s)]^2 ds]$. Although this can be relaxed somewhat in order to define the stochastic integral, the mean of the integral may cease to exist or may exist and be different from 0 if $E[\int_0^T [F(s)]^2 ds] = \infty$