6. Transition to Continuous Space. Random Walks and the Heat equation.

Consider the following Markov Process on the integers $Z = \{i : -\infty < i < \infty\}$. The process x_n starts from 0 at time $n = 0$ and at each step moves one unit to the right with probability p or one unit to the left with probability $q = 1 - p$. The choices at successive steps are made independently, but always with the same probabilities p and q for moving right or left. This clearly defines a Markov Process on Z with

$$\pi_{i,i+1} = p; \quad \pi_{i,i-1} = q; \quad \text{and} \quad \pi_{i,j} = 0 \quad \text{for} \quad |i - j| \neq 1$$

the n step transition probabilities are easy to calculate.

$$\pi_{i,j}^{(n)} = 0 \quad \text{unless} \quad |j - i| \leq n \quad \text{and} \quad j - i = n \pmod 2$$
$$\pi_{i,j}^{(n)} = \binom{n}{r} p^r q^{n-r} \quad \text{where} \quad 2r - n = j - i$$

One can obtain this formula by noting that out of the n steps r were to the right and the reamaining $n - r$ were to the left $x_n = i + r - (n - r) = i + 2r - n$ and $x_n = j$ if $j - i = 2r - n$. this is possible only if $j - i = n$ (mod 2), and the probability is given by the binomial distribution. The law of large numbers for the binomial says that for large n the probability is nearly 1 that $\frac{r}{n}$ is close to p. More precisely, for any $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{r:|\frac{r}{n}-p|\leq\epsilon} \binom{n}{r} p^r q^{n-r} = 1$$

This means that for any bounded continuous function f on R

$$\lim_{n\to\infty} E[f(\frac{x_n}{n})|x_0 = 0] = f(p - q)$$

If we denote by P the operator

$$(Pu)(i) = pu(i + 1) + (1 - p)u(i - 1)$$

Our statement concerns the function $u_n(i) = f(\frac{i}{n})$ and the claim is

$$\lim_{n\to\infty} (P^n u_n)(0) = p - q$$

Let us see if we can figure out why this is true. Let us rescale sapce and time by a step size of $h = \frac{1}{n}$. Then

$$(Pu)(x) = pu(x + h) + qu(x - h)$$

Let us define $u(kh, x) = (P^k u)(x)$. We see that

$$u((k+1)h, x) = pu(kh, x+h) + qu(kh, x-h)$$

Or

$$\frac{1}{h}[u((k+1)h, x) - u(kh, x)] = \frac{1}{h}p[u(kh, x+h) - u(kh, x)] + \frac{1}{h}q[u(kh, x-h) - u(kh, x)]$$

Passing to the limit as $h \to 0$ we get

$$u_t = (p-q)u_x$$

The solution with $u(0, x) = f(x)$ is given by

$$u(t, x) = f(x + (p-q)t)$$

We are interested in $u(1, 0)$ which is $f(p-q)$. The law of large numbers for the binomial is just the approximation of a first order PDE by difference equations.

Let us now consider the case where $p = q = \frac{1}{2}$. Now $P^n u_n(x) \to f(x)$, so nothing much happens. The space scale has to be $\frac{1}{\sqrt{n}}$ to get something significant. In the probabilistic setting we are looking at $\frac{x_n}{\sqrt{n}}$ which satisfies the central limit theorem and the correct behavior is with $u(x) = f(\frac{x}{\sqrt{n}})$

$$(P^n u_n)(0) \to \int \frac{1}{\sqrt{2\pi}} f(y) e^{-\frac{y^2}{2}} dy$$

This is just as before except we get

$$\frac{1}{h}[u((k+1)h, x) - u(kh, x)] = \frac{1}{2h}[u(kh, x+\sqrt{h}) - u(kh, x)] + \frac{1}{2h}q[u(kh, x-\sqrt{h}) - u(kh, x)]$$

and passing to the limit as $h \to 0$, we get the heat equation

$$u_t = \frac{1}{2}u_{xx} \tag{1}$$

with $u(0, x) = f(x)$. The solution is given by

$$u(t, x) = \int \frac{1}{\sqrt{2\pi t}} f(y) e^{-\frac{(y-x)^2}{2t}} dy$$

and $u(1, 0)$ is then

$$\int \frac{1}{\sqrt{2\pi}} f(y) e^{-\frac{y^2}{2}} dy$$

In other words, the central limit theorem for the binomial (with $p = q = \frac{1}{2}$) can be interpreted as the convergence of the solution of

$$u(t+h,x) = \frac{1}{2}[u(t,x+\sqrt{h}) + u(t,x-\sqrt{h})]$$

to the corresponding solution of the heat equation (1).

We can have time varying continuously, and consider a Markov process on Z with transition rates

$$a_{i,i+1} = a_{i,i-1} = \frac{1}{2};\ \ a_{i,i} = -1;\quad \text{and}\quad a_{i,j} = 0\ \ \text{otherwise.}$$

The expectation

$$u(t,i) = E[f(x(t)|x(0) = i]$$

will satisfy

$$\frac{du(t,i)}{dt} = \frac{1}{2}[u(t,i+1) + u(t,i-1) - 2u(t,i)];\quad u(0,i) = f(i)$$

If we rescale space by $\sqrt{h}$ and time by h, the equations become

$$\frac{du(t,x)}{dt} = \frac{1}{2h}[u(t,x+\sqrt{h}) + u(t,x-\sqrt{h}) - 2u(t,x)];\quad u(0,x) = f(x)$$

which again converges to the solution of the heat equation. What we have is again a central limit theorem for the distribution of $\frac{x(t)}{\sqrt{t}}$ as $t \to \infty$.

Examples.

1. We will show that the distribution of $x(t)$ is the distribution of the diffrence $X_1 - X_2$ of two independent Poisson random variables with parameters $\frac{t}{2}$. Let us try $f(i) = e^{\lambda i}$. The solution $u(t,i)$ can be obtained by separation of variables. Set $u(t,i) = e^{\lambda i} g(\lambda, t)$. Then,

$$\frac{dg}{dt} = \frac{1}{2}[e^{\lambda} + e^{-\lambda} - 2]g;\quad g(\lambda, 0) = 1$$

will do it. This gives

$$g(\lambda,t) = \exp[\frac{t}{2}[(e^{\lambda} - 1) + (e^{-\lambda} - 1)]] = E[e^{\lambda(X_1 - X_2)}]$$

with X_1, X_2 having independent Poisson distributions with parameter $\frac{t}{2}$.

Suppose $\pi_h(x,y)dy$ is the transition density of a Markov chain, with h representing the time step. Let us make the following assumptions:

$$\sup_x |\int (y-x)\pi_h(x,y)dy| = o(h)$$
$$\sup_x |\int (y-x)^2\pi_h(x,y)dy - h| = o(h)$$
$$\sup_x \int (y-x)^4\pi_h(x,y)dy = o(h)$$

Then the function

$$u_h(n,x) = E[f(x_n)|x_0 = x]$$

converges to the solution u of the heat equation (1) provided $nh \to t$.

Proof: Let us start with the solution $u(t,x)$ of the heat equation which we will assume is a smooth function of t and x. Let

$$\Delta_k = E[u(t-(k+1)h, x_{k+1})|x_0 = x] - E[u(t-kh, x_k)|x_0 = x]$$

Assuming $nh = t$, this is a telescoping sum and

$$\sum_{k=0}^{n-1} \Delta_k = u_h(n,x) - u(t,x)$$

It is therefore sufficient to prove that each

$$\Delta_k = o(h)$$

If we expand by Taylor's formula

$$\begin{aligned} u(t-(k+1)h, y) - u(t-kh, x) &= -hu_t(t-kh, x) + u_x(t-kh, x)(y-x) \\ &\quad + \frac{1}{2}u_{xx}(t-kh, x)(y-x)^2 + \text{ Remainder} \end{aligned}$$

We can estimate

$$E[u(t-(k+1)h, x_{k+1}) - u(t-kh, x_k)|x_k = x]$$

by $o(h)$, because upon integrating with $\pi_h(x,y)$, with errors that are $o(h)$, we get $h(u_t - \frac{1}{2}u_{xx}) = 0$. The remainder term can be estimated by

$$\begin{aligned} \int |y-x|^3\pi_h(x,y)dy &\le (\int |y-x|^2\pi_h(x,y)dy)^{\frac{1}{2}}(\int |y-x|^4\pi_h(x,y)dy)^{\frac{1}{2}} \\ &\le o(h) \end{aligned}$$

To go from $E[Q|x_k]$ to $E[Q|x_0]$ is easy because the conditional expectation of anything that is $o(h)$ is still $o(h)$.

Remarks:

1. We have assumed that $\pi_h(x,y)dy$ is given by a density only for convenience. In principle $\pi_h(x,\cdot)$ is just the probability distribution of x_1 given $x(0)=x$, and does not have to be given by a density. It can be a discrete distribution as well.

2. The first two assumptions say that to with in $o(h)$ the infinitesimal 'mean' is 0 and the infinitesimal 'variance' is h.

3. The third condition is important and needs to be understood. A random variable X with mean 0 and variance h can come in different shapes. For example X can be $\pm\sqrt{h}$ with probability $\frac{1}{2}$ each. Or X can be 0 with probability $1-h$ and ± 1 with probability $\frac{h}{2}$ each. In both cases the mean and variance check out. However $E[X^4]=h^2$ in the first case and h in the second. The clue is in the calculation of

$$\lim_{h\to 0}\frac{1}{h}E[f(X)-f(0)]=\frac{1}{2}f_{xx}(0)$$

in the first case and

$$\lim_{h\to 0}\frac{1}{h}E[f(X)-f(0)]=\frac{1}{2}[f(1)+f(-1)-2f(0)]$$

in the second.

Examples:

2*. (Compare with Ex 3* of section 2). Suppose $\pi_h(x,y)$ satisfies

$$\sup_x\left|\int(y-x)\pi_h(x,y)dy-hb(x)\right|=o(h)$$

and

$$\sup_x\left|\int(y-x)^2\pi_h(x,y)dy\right|=o(h)$$

for some nice smooth bounded function $b(x)$, prove the 'law of large numbers'

$$\lim_{\substack{h\to 0\\ n\to\infty\\ nh\to t}} P[|x_n-g(t)|\geq\epsilon|x_0=x]\to 0$$

where $g(t)$ is the value of the solution at time t of

$$\frac{dg}{ds}=b(g(s));\ g(0)=x$$

Note that it is sufficient to prove

$$\lim_{\substack{h\to 0\\ n\to\infty\\ nh\to t}} u_h(n,x)=\lim_{\substack{h\to 0\\ n\to\infty\\ nh\to t}} E[f(x_n)|x_0=x]=u(t,x)=f(g(t))$$

where u solves $u_t=b(x)u_x$ with $u(0,x)=f(x)$.