5. Continuos time models with discrete state space and systems of ODEs.

The transition probabilities $\pi_{i,j}^{(n)}$ are all obtained as entries of the n-th power P^n starting from one stochastic matrix P. If we are dealing with transitions in continuous time, we need to consider $P(t) = \{\pi_{i,j}(t)\}$, defined for $t > 0$.

$$\pi_{i,j}(t) = P[x(s+t) = j | x(s) = i] \quad \text{for all} \quad i, j, s \geq 0, t > 0$$

From the Markov property

$$\begin{aligned}\pi_{i,j}(t+s) &= P[x(t+s) = j | x(0) = i] \\ &= \sum_k P[x(t) = k, x(t+s) = j | x(0) = i] = \sum_k \pi_{i,k}(t)\, \pi_{k,j}(s)\end{aligned}$$

or $P(t+s) = P(t)P(s)$. Since the probability of changing to a new state in a short time is small, we expect $P(t) \to I$ as $t \downarrow 0$. It is a fact that such matrix valued functions are differentiable and if $P'(0) = A$, then

$$P'(t) = A\, P(t) = P(t)\, A$$

and one can write

$$P(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

The matrix $A = \{a_{i,j}\}$ has the following properties:

$$a_{i,i} \leq 0, \;\; a_{i,j} \geq 0 \;\; \text{for} \;\; i \neq j, \;\; \sum_j a_{i,j} = 0$$

The entries $a_{i,j}$ for $i \neq j$ have the interpretation

$$\pi_{i,j}(t) = a_{i,j} t + o(t) \;\; \text{as} \;\; t \to 0$$

and

$$\pi_{i,i}(t) - 1 = -\sum_{j \neq i} \pi_{i,j}(t) = -\sum_{j \neq i} a_{i,j} t \; + o(t) = a_{i,i} t + o(t)$$

If we consider $u(t) = P(t) f$, then

$$u'(t) = P'(t)\, f = AP(t)\, f = A\, u(t)$$

or

$$\frac{du_i(t)}{dt} = \sum_j a_{i,j} u_j(t); \quad u(i,0) = f(i) \tag{1}$$

Denoting $u_i(t)$ more conveniently as $u(t,i)$, it is is not hard to see that

$$u(t,i) = E[f(x(t))|x(0) = i] \tag{2}$$

The process $x(t)$ is of course a Markov process such that

$$P[x(t+s) = j|x(\sigma) : 0 \leq \sigma \leq s] = \pi_{x(s),j}(t) = P[x(t+s) = j|x(s)]$$

If $p_i = P[x(0) = i]$, then

$$P[x(0) = i_0, x(t_1) = i_1, \ldots, x(t_n) = i_n] = p_{i_0}\pi_{i_0,i_1}(t_1)\cdots\pi_{i_{n-1},i_n}(t_n - t_{n-1})$$

Since the solution of a system ODE's is uniquely determined by its initial values, the expecation in (2) is the unique solution of (1). Some times we need to calculate expecations of the form

$$u(t,i) = E[f(x(t)) + \int_0^t g(x(s))ds|x(0) = i]$$

This is a solution of

$$\frac{du(t)}{dt} = A\,u(t) + g; \quad u(0) = f$$

or

$$\frac{du(t,i)}{dt} = \sum_j a_{i,j}\,u(t,j) + g(i); \quad u(0,i) = f(i) \ \text{ for } \ 1 \leq i \leq N$$

It is not hard to prove.

$$u(t) = P(t)f + \int_0^t P(s)gds = P(t)f + \int_0^t P(t-s)gds$$

and

$$u'(t) = P'(t)f + P(t)g = AP(t)f + g + A\int_0^t P(t-s)\,g\,ds = Au(t) + g$$

Examples:

1. Let us consider a system with two states 1 and 2 with $a_{1,1} = -1, a_{1,2} = 1, a_{2,1} = a_{2,2} = 0$. It is easy to see that

$$\pi_{1,1}(t) = e^{-ct}, \pi_{1,2}(t) = 1 - e^{-ct}, \pi_{2,1}(t) = 0, \pi_{2,2}(t) = 1$$

The system starting from 1 waits for a random time τ with $P[\tau \geq t] = e^{-ct}$ and then jumps to 2. It stays there for ever. The moral is that waiting times have exponential distributions.

2. We can make the state space countable $0, 1, \ldots\ldots$ with $a_{i,i} = -c$ and $a_{i,i+1} = c$ for all $i \geq 0$. The system waits for an exponential time with parameter c and moves one steps to the right. One can check again that

$$\pi_{i,j}(t) = e^{-ct} \frac{(ct)^{j-i}}{(j-i)!} \quad \text{for} \quad j \geq i; \ \pi_{i,j}(t) = 0 \quad \text{for} \quad j < i$$

This is the basic Poisson process with rate c.

$$(Au)(i) = c[u(i+1) - u(i)]$$

If we make the state space $\{jh\}$ by rescaling space and speed up the rate to $\frac{c}{h}$, and if $u(x)$ is a smooth function on $[0, \infty)$, then

$$\lim_{h \to 0} (A_h u)(x) = cu'(x)$$

suggesting that

$$u_h(t, x) \to f(x + ct)$$

which solves

$$u_t = cu_x; \quad u(0, x) = f(x)$$

We are back to a difference approximation for an ODE.

3*. For the two state case with $a_{1,2} = a_{2,1} = 1, a_{1.1} = a_{2,2} = -1$, calculate $\pi_{i,j}(t)$ for $i, j = 1, 2$.

Discounted Integrals. Just as in the discrete case we will try to evaluate

$$u(i) = E\left[\int_0^\infty e^{-ct} f(x(t)) dt | x(0) = i\right]$$

Using $AP(t) = P'(t)$ and integrating by parts,

$$Au = A \int_0^\infty e^{-ct} P(t) f dt = \int_0^\infty e^{-ct} P'(t) f dt = -f + c \int_0^\infty e^{-ct} P(t) f dt = -f + cu$$

Or u solves $cu - Au = f$. Formally, since $P(t) = e^{tA}$,

$$\int_0^\infty e^{-ct} e^{tA} dt = (cI - A)^{-1}$$

The countable case can lead to problems of uniqueness. If the jump rate c_i, from $i \to i+1$ grows too fast so that $\sum_i c_i^{-1} < \infty$, then the system will jump itself to infinity in finite time. Since there will be positive probability that this hppens before time t, the solution to the infinite system of ODE's will not be unique.

4. Can you make this precise? Take $\tau_1, \tau_2, \ldots$ to be a sequence of independent exponential random variables with $E\tau_k = \frac{1}{c_k}$ and $\sum_k \frac{1}{c_k} < \infty$ Then

$$u_i(t) = P[\tau_{i+1} + \cdots \geq t]$$

will be a soultion of

$$\frac{du_i(t)}{dt} = c_i[u_{i+1}(t) - u_i(t)] \tag{3}$$

with $u_i(0) = 1$. But $u_i(t) \to 0$ as $t \to \infty$ and is different from $u_i(t) \equiv 1$ which is another solution.

Hint: Truncate at level N. Let the system come to rest at state N. Then

$$u_i^N(t) = P[\tau_{i+1} + \cdots + \tau_N \geq t] = P[x(t) \neq N | x(0) = i] = E[f(x(t))|x(0) = i]$$

where $f(i) = 1$ if $i < N$ and $f(N) = 0$. Now $u_i^N(t)$ satsfies (3). Pass to the limit as $N \to \infty$.

Feynman-Kac formula. We try to compute

$$u(t,i) = E\left[\exp[\int_0^t g(x(s))ds] f(x(t)) | x(0) = i\right]$$

as the solution of a system of ODE's. Clearly

$$u(t) = Q(t)f$$

for some $Q(t)$. By Markov property,

$$u(t+s,i) = E\left[\exp[\int_0^t g(x(s))ds] u(s, x(t))) | x(0) = i\right]$$

Therefore $Q(t+s) = Q(t)Q(s)$. Therefore if $Q'(0) = B$, $u(t)$ will satisfy

$$u'(t) = Bu(t); u(0) = f$$

We only need to compute B. For small t

$$u(t,i) = E[f(x(t)|x(0) = i] + E[f(x(t)) \int_0^t g(x(s))ds | x(0) = i] + O(t^2)$$

Therefore

$$Bf = Af + gf$$

which is the Feynman-Kac formula.

Example:

5*. The inflation rate can be in one of two states; a low rate of 3% and a high one of 8%. The economy shifts randomly between the two states at rate of 1. The unit of time is irrelevent and can be taken as year. If one has an infinite steady future income stream of $1 per year what is the expected discounted present value of the total income. Calculate it for each of the two present states of the economy.

Continuous Parameter Martingales:. Just as in the discrete case, a random process $Z(t)$ given for $t \geq 0$, that depends on the past history $x(s) : 0 \leq s \leq t$ of the process $x(\cdot)$ is called a martingale if

$$E[Z(t)|x(\sigma) : 0 \leq \sigma \leq s] = Z(s) \quad \text{for} \quad 0 \leq s \leq t$$

The following are examples of martingales.

$$f(x(t)) - f(x(0)) - \int_0^t [Af](x(s)ds$$
$$u(t, x(t)) - u(0, x(0)) - \int_0^t [u_s(s, x(s)) + (Au)(s, x(s))ds$$

Roughly, it is enough to check that

$$E[f(x(h)) - f(x(0)) - \int_0^h [Af](x(s))ds|x(0) = i] = o(h)$$

which is easy because, basically that was how A was defined. The main step is

$$E[f(x(h))|x(0) = i] = f(i) + h(Af)(i) = o(h)$$

Then we write

$$f(x(t) - f(x(0)) - \int_0^t (Af)(x(s))ds$$

as the sum of roughly $\frac{t}{h}$ terms

$$\sum_j \left[f(x((j+1)h)) - f(x(jh)) - \int_{jh}^{(j+1)h} (Af)(x(s))ds \right]$$

Since the expectation of each term is $o(h)$ we are done.