4. Transition Matrices for Markov Chains. Expectation Operators.

Let us consider a system that at any given time can be in one of a finite number of states. We shall identify the states by $\{1, 2, \ldots, N\}$. The state of the system at time n will be denoted by x_n. The system is 'noisy' so that $x_n = F(x_{n-1}, \xi_n)$, where $\{\xi_j : j \geq 1\}$ are independent random variables with a common distribution. We saw before that the important quantities are

$$\pi_{i,j} = P[\xi : F(i, \xi) = j]$$

The matrix $\{\pi_{i,j}\}$, $1 \leq i, j \leq N$, has the following properties.

$$\pi_{i,j} \geq 0 \quad \text{for all} \quad 1 \leq i, j \leq N \tag{1}$$

and

$$\sum_{j=1}^{N} \pi_{i,j} = 1 \quad \text{for all} \quad 1 \leq i \leq N. \tag{2}$$

Such a matrix is called a transition probability matrix and defines the probability of transition from state i to state j in one step. Since a matrix represents a linear transformation on the vector space R^n with points $\bar{f} = \{f_1, \ldots, f_n\}$, the matrix $P = \{\pi_{i,j}\}$ is equivalently given by the linear trnasformation

$$\bar{g} = P\ \bar{f}$$

or

$$g_i = \sum_{j=1}^{N} \pi_{i,j}\, f_j \quad \text{for} \quad 1 \leq i \leq N$$

The properties (1) and (2) translate into $P\,\bar{f} \geq 0$ if $\bar{f} \geq 0$ and $P\,\bar{1} = \bar{1}$. Here by $\bar{f} \geq 0$ we mean $f_i \geq 0$ for each i and by $\bar{1}$ we mean the vector $(1, \ldots, 1)$. Such matrices are called stochastic matrices. the linear transformation P acting on vectors has a natural interpretation. A vector $\bar{f}$ can be thought of as a function defined on the state space $\mathcal{S} = \{\infty, \ldots, \mathcal{N}\}$, with $f(i) = f_i$. Then

$$g(i) = E[f(x_n)|x_{n-1} = i] = E[f(F(i, \xi_n))] = \sum_{j=1}^{N} f(j)\, \pi_{i,j} = (P\bar{f})_i \tag{3}$$

or

$$\bar{g} = P\bar{f}$$

is the conditional expectation operator.

The transpose of P which acts on vectors $\bar{p}$ by

$$q_j = \sum_{i=1}^{N} \pi_{i,j}\, p_i \tag{4}$$

has an interpretation too. If at time n, the system can be in the state i with probbaility p_i, the probability of finding it in state j at time $n+1$ is $q_j = \sum_i p_i \pi_{i,j}$. In other words P acting on the left as $\bar{q} = \bar{p}P$ propagates probabilities. One can see this duality by the simple calculation

$$\sum f(j)\, q_j = E[f(x_{n+1})] = E[E[f(x_{n+1})|x_n]] = E[g(x_n)] = \sum_i g(i)\, p_i$$

or

$$< \bar{q}, \bar{f} >=< \bar{p}P\bar{f} >=< \bar{p}, \bar{g} >= \sum_{i,j} \pi_{i,j} p_i f(j)$$

Multi-step transitions: If we assume that the process is Markov, then if at time 0 the initially the system can be in state i with probability p_i, then the probability of finding the system successively in states $x_0, x_1, \ldots, x_n$ at times $0, 1, \ldots, n$ respectively is equal to

$$p(x_0, x_1, \ldots, x_n) = p_{x_0} \pi_{x_0,x_1} \cdots \pi_{x_{n-1},x_n}$$

and gives us the joint distribution of $x_0, x_1, \ldots, x_n$. By a simple calculation

$$E[f(x_n)|x_0] = \sum_{x_1,x_2,\ldots,x_n} f(x_n)\ \pi_{x_0,x_1} \cdots \pi_{x_{n-1},x_n} = \sum_j \pi^{(n)}_{x_0,j} f(j)$$

where

$$\pi^{(n)}_{i,j} = \sum_{x_1,x_2,\ldots,x_{n-1}} \pi_{i,x_1} \cdots \pi_{x_{n-1},j}$$

These are the n-step transition probabilities and can be defined inductively by

$$\pi^{(n)}_{i,j} = \sum_k \pi_{i,k} \pi^{(n-1)}_{k,j} = \sum_k \pi^{(n-1)}_{i,k} \pi_{k,j}$$

One can see that this is the same as

$$\pi^{(n)}_{i,j} = (P^n)_{i,j}$$

In other words the powers of the matrix P provide the transition probabilities for multiple steps.

Examples:

1. Consider the case where there are just two states with transition probabilities $\pi_{1,1} = \pi_{2,2} = p$ and $\pi_{1,2} = \pi_{2,1} = q = 1 - p$. It is clear that $\pi_{1,1}^{(n)} = \pi_{2,2}^{(n)} = p_n$ and $\pi_{1,2}^{(n)} = \pi_{2,1}^{(n)} = q_n = 1 - p_n$. More over p_n satisfies

$$p_n = pp_{n-1} + q(1 - p_{n-1}) = (p - q)p_{n-1} + q$$

We can solve it to get

$$p_n = \frac{1}{2}(p - q)^n + \frac{1}{2}$$

2*. If $\pi_{1,1} = p$, $\pi_{1,2} = q = 1 - p$, $\pi_{2,1} = 0$ and $\pi_{2,2} = 1$, calculate $\pi_{i,j}^{(n)}$ for $i, j = 1, 2$.

Discounted sums:. Let the system evolve from a state i. For each n, if the system at time n is in state j a reward of $f(j)$ is obtained. There is a discount factor $0 < \rho < 1$ so that the total reward is

$$\sum_{n=0}^{\infty} \rho^n f(x_n)$$

One is interested in the calculation of the expected discounted total reward

$$E\left[\sum_{n=0}^{\infty} \rho^n f(x_n)\middle| x_0 = i\right] = g(i)$$

The numbers $g(i)$ can be obtained by solving the nonsingular system of linear equations

$$(I - \rho P)g = f$$

or

$$g(i) - \rho \sum_{j=1}^{N} \pi_{i,j} g(j) = f(i)$$

for $i = 1, \ldots, N$.

We will provide several ways of proving it.

First note that

$$g(i) = E\left[\sum_{n=0}^{\infty} \rho^n f(x_n)\middle| x_0 = i\right] = \sum_{n=0}^{\infty} \rho^n (P^n f)(i) = ((I - \rho P)^{-1} f)(i)$$

by computing the expectation term by term.

This can be justified by the computation

$$(1-\rho P)g = (1-\rho\, P)\sum_{n=0}^{\infty}\rho^n\ P^n\, f = \sum_{n=0}^{\infty}\rho^n\ P^n\, f - \sum_{n=1}^{\infty}\rho^n\ P^n\, f = f$$

The matrix $(I-\rho P)$ is seen to be invertible if $0 \le \rho < 1$.

Another way of deriving the equation is to write the sum

$$S = \sum_{n=0}^{\infty}\rho^n\, f(x_n)$$

as

$$f(x_0) + \rho\sum_{n=0}^{\infty}\rho^n f(x_{n+1})$$

If at time 1 we are at $x_1 = j$, then

$$E\left[\sum_{n=0}^{\infty}\rho^n\, f(x_{n+1})\middle| x_1 = j\right] = g(j)$$

In other words

$$g(i) = f(i) + \rho\, E[g(x_1)|x_0 = i] = f(i) + \rho\,\sum_j \pi_{i,j} g(j)$$

or

$$(I-\rho\, P)g = f$$

A third approach uses the notion of martingales. A collection $\{u_n(x_0, x_1, \ldots, x_n)\}$ of functions depending on the history $(x_0, x_1, \ldots, x_n)$ is called a martingale if

$$E[u_n(x_0, x_1, \ldots, x_n)|x_0, \ldots, x_{n-1}] = u_{n-1}(x_0, x_1, \ldots, x_{n-1})$$

If $\{u_n\}$ is a martingale, then

$$E[u_n|x_0] = u_0(x_0)$$

Consider

$$u_n(x_0, \ldots, x_n) = \rho^n g(x_n) + \sum_{k=0}^{n-1}\rho^k f(x_k)$$

Then

$$E[u_n(x_0, x_1, \ldots, x_n)|x_0, \ldots, x_{n-1}] = \rho^n\,(P\, g)(x_{n-1}) + \sum_{k=0}^{n-1}\rho^k f(x_k)$$

If $\rho\, Pg = g - f$, we get

$$\begin{aligned} E[u_n(x_0, x_1, \ldots, x_n)|x_0, \ldots, x_{n-1}] &= \rho^{n-1}[g-f](x_{n-1}) + \sum_{k=0}^{n-1} \rho^k f(x_k) \\ &= u_{n-1}(x_0, x_1, \ldots, x_{n-1}) \end{aligned}$$

Therefore u_n is a martingale and its expectation is constant. Equating expectations at $n = 0$ and $n = \infty$, we get

$$g(x_0) = u_0(x_0) = E\left[\sum_{n=0}^{\infty} \rho^n f(x_n) \middle| x_0\right]$$

Example :

3.* In example 1, consider the function f defined by $f(1) = 1$ and $f(2) = 0$. Evaluate

$$g(i) = E\left[\sum_{n=0}^{\infty} \rho^n f(x_n) \middle| x_0 = i\right]$$

by direct calculation as well as by solving the equation described above.