14. Optimal Stopping.

Consider the simple problem of optimal stopping of a one dimensional Brownian Motion. We have a function f and wish to stop at a stopping time τ such that

$$E\big[e^{-\tau}f(x(\tau))\big]$$

is a maximum. Suppose the optimal value is $u(x)$ as a function of the staring point $x = x(0)$. Clearly, since one particular option is to stop right away $u(x) \geq f(x)$. It is clear that, if $u(x) > f(x)$ we should not stop right away and if we reach the set where $u(x) = f(x)$ we should stop. If we do not stop for time h, the Brownian motion will be at some point $x(h)$ and the optimal value now will be $u(x(h))e^{-h}$. Sinec this may be suboptimal, It follows that

$$u(x) \geq E_x\left[u(x(h))e^{-h}\right]$$

This implies

$$\frac{1}{2}u_{xx} - u \leq 0$$

and on the continuation set where $u(x) > f(x)$ we have equality in the equation above. On the stopping set, since $f = u$, we must have $\frac{1}{2}f_{xx} - f \leq 0$.

Example: Let us take $f(x) = x^2$. Then the stoping set is contained in $1 - x^2 \leq 0$ or $|x| \geq 1$. A continuation set of the form $|x| < a$ with $a \geq 1$ makes sense. We stop when we reach $|x| > a$, the value at x is given by solving $\frac{u_{xx}}{2} - u = 0$, with $u(\pm a) = a^2$, for $|x| \leq a$. This can be explicitly solved by $u(x) = a^2\frac{\cosh\sqrt{2}x}{\cosh\sqrt{2}a}$. The value of a is to be deteremined. This is done by matching derivatives $u'(\pm a) = f'(\pm a) = \pm 2a$.

$$a^2\sqrt{2}\frac{\sinh\sqrt{2}a}{\cosh\sqrt{2}a} = 2a$$

or

$$\tanh\sqrt{2}a = \frac{\sqrt{2}}{a}$$

There is a unique $a_0 > 0$ such that $\pm a_0$ are the solutions. Since $|\tanh x| < 1$ we must have $a_0 > \sqrt{2} > 1$. Finally we must show that $u(x) \geq x^2$ on $|x| \leq a_0$. Note that $u - f = h$ has the property $h(\pm a_0) = 0$. If $h < 0$ some where then h must have a minimum in the interior. At that point b with $|b| < a_0$, we must have $h'(b) = 0$ or $u'(b) = f'(b)$. Only such point is a_0 and we are done. let us now prove that

$$u(x) = \sup_{\tau} E_x\left[e^{-\tau}f(x(\tau)\right]$$

Since $\frac{1}{2}u_{xx} - u \leq 0$, is piecewise smooth and and continuously differentiable, by Itô's formula, $u(x(t))e^{-t}$ is a suermartingale and therefore

$$u(x) \geq E_x\left[e^{-\tau}f(x(\tau)\right]$$

for any τ. On the other hand for $|x| < a_0$, $\tau_0 = \inf\{t : |x(t)| = a_0$ yields a value $u(x)$. For $|x| > a_0$, $\tau = 0$ works and $u = f$.

Perpetual American Put.

Let us consider a Black and Scholes model with a geometric Brownian motion

$$dx(t) = \sigma x(t) d\beta(t)$$

with a discount rate r. Assuming a strike price of A, the payoff is

$$f(x(t)) = (A - x(t))^+$$

The optimal policy clearly is going to be to exercise the option as soon as $x(t) \leq E$ for some $E \leq A$. We solve the equation and find E, such that

$$\frac{\sigma^2 x^2}{2} u_{xx} + r x u_x - r u = 0$$

with $u(E) = (A - E)$ and $u'(E) = -1$. Trying $u(x) = x^{-a}$, we need

$$\sigma^2 \frac{a(a+1)}{2} - ar - r = 0$$

or

$$a = \frac{2r}{\sigma^2}$$

With a solution

$$u(x) = cx^{-a}$$

$$cE^{-a} = A - E \ ; \qquad caE^{-(a+1)} = 1$$

Solve for E and c.

$$\frac{E}{a} = A - E$$

There is a solution with

$$E = \frac{a}{a+1} A, \quad c = c(a, A) = \frac{a^a A^{a+1}}{(a+1)^{(a+1)}}$$

This can be seen as an arbitrage value. If τ is the exercise time, by Itô's formula

$$\begin{aligned} e^{-r\tau} u(x(\tau)) = u(x) &+ \int_0^\tau e^{-rt} u_x(x(s)) dx(s) \\ &+ \frac{\sigma^2}{2} \int_0^\tau e^{-rt} x^2(s) u_{xx}(x(s)) ds - r \int_0^\tau e^{-rt} u(x(s)) ds \\ = u(x) &+ \int_0^\tau e^{-rt} u_x(x(s))[dx(s) - r x(s) ds] \end{aligned}$$

On the other hand if at time t the option is not exercised while the stock has been below E,

$$\begin{aligned}
e^{-rt}u(x(t)) &= u(x) + \int_0^t e^{-rs}u_x(x(s))dx(s) \\
&\quad + \frac{\sigma^2}{2}\int_0^t e^{-rs}x^2(s)u_{xx}(x(s))ds - r\int_0^t e^{-rs}u(x(s))ds \\
&= u(x) + \int_0^t e^{-rs}u_x(x(s))[dx(s) - rx(s)ds] \\
&\quad - rA\int_0^t \mathbf{1}_{[0,E]}(x(s))e^{-rs}ds
\end{aligned}$$

and the last term is a profit for the option writer.

Optimal stopping with a finite time horizon.

The problem of determining the optimal stopping time is much more complicated now because the value function is a function of both space and time. If for Browninan motion $x(t)$ we want to optimize

$$\sup_{\tau \leq T} E\left[f(t, x(t))\right]$$

the value is the function

$$u(t, x) = \sup_{t \leq \tau \leq T} E\left[f(t, x(t)) | x(t) = x\right]$$

the function $u(t, x)$ will satisfy $u(t, x) \geq f(t, x)$ as well as

$$u_t + \frac{1}{2}u_{xx} \leq 0$$

with equality holding on the set $u(t, x) > f(t, x)$. One can abstractly characterize u as

$$u(t, x) = \inf\left\{v(t, x) : v_t + \frac{1}{2}v_{xx} \leq 0, v(t, x) \geq f(t, x)\right\}$$

This is just a staraight forward consequence of the definition. In parctical terms, there will be two regions in (x, t) space with a boundary between them (hopefully a smooth curve). On one side, the stopping side where $u = f$, we will have $f_t + \frac{f_{xx}}{2} \leq 0$. On the other side we will have $u(t, x) > f(t, x)$, $u_t + \frac{u_{xx}}{2} = 0$ with $u = f$ and $u_x = f_x$ all along the boundary.

Some times, the optimal value $u(t, x)$ is seen to be of special form, like

$$u(t, x) = (T - t)^a \phi(\frac{x}{\sqrt{T - t}})$$

for instance. In such cases the problem can be reduced to a one dimensional problem and more or less explicit solutions can be found. Consider for instance

$$f(x, t) = x^4 + 3(T - t)^2$$

with a terminal time T and a starting time $t < T$. It is clear by Browninan scaling that we should try

$$u(t,x) = (T-t)^2 \phi(\frac{x}{\sqrt{T-t}})$$

The boundary would be a parabola $x^2 = C(T-t)$, with C to be determined.

$$u_t + \frac{u_{xx}}{2} = 0$$

becomes

$$-2(T-t)\phi(\frac{x}{\sqrt{T-t}}) + (T-t)^2\phi'(\frac{x}{\sqrt{T-t}})\frac{x}{2(\sqrt{T-t})^3} + \frac{T-t}{2}\phi''(\frac{x}{\sqrt{T-t}}) = 0$$

or

$$-2\phi(\frac{x}{\sqrt{T-t}}) + (T-t)\phi'(\frac{x}{\sqrt{T-t}})\frac{x}{2(\sqrt{T-t})^3} + \frac{1}{2}\phi''(\frac{x}{\sqrt{T-t}}) = 0$$

Reduces to

$$\frac{1}{2}\phi''(y) + \frac{y}{2}\phi'(y) - 2\phi(y) = 0$$

Try

$$\phi(y) = y^4 + ay^2 + b$$

$$6y^2 + a + 2y^4 + ay^2 - 2[y^4 + ay^2 + b] = 0$$

The choice of $a = 6$, $b = 3$ works. Since $f = (T-t)^2 g(\frac{x}{\sqrt{T-t}})$ with $g(y) = y^4 + 3$, we need to solve for

$$y^4 + 3 = c(y^4 + 6y^2 + 3), 4y^3 = c(4y^3 + 12y)$$

replacing y^2 by C,

$$C^2 + 3 = c(C^2 + 6C + 3), C = cC + 3c$$

Since $c = \frac{C}{C+3}$, we slove for

$$(C^2 + 3)(C + 3) = C(C^2 + 6C + 3)$$

or

$$3C^2 + 3C + 9 = 6C^2 + 3C$$

giving us

$$C = \sqrt{3}, c = \frac{1}{1+\sqrt{3}}$$

All we have to decide is what problem did we solve? Did we maximize or minimize? Of the two regions $x^2 \le C(T-t)$ and $x^2 \ge C(T-t)$ which is the contnuation region and which is the stopping region? Since $f_t + \frac{f_{xx}}{2} = 6[x^2 - (T-t)]$ is negative near zero and positive far out, if we want to maximize the continuation region should definitely include $x^2 > (T-t)$ and to minimize it should similarly contain $x^2 < (T-t)$. Our regions are $x^2 \ge \sqrt{3}(T-t)$ and $x^2 \le \sqrt{3}(T-t)$. The first choice contains neither, while the second contains $x^2 < (T-t)$. Therefore the continuation region is $x^2 \leq \sqrt{3}(T-t)$, we are minimizing and

$$u(t,x) = \frac{1}{1+\sqrt{3}}[x^4 + 6x^2(T-t) + 3(T-t)^2]$$

represents the value in that region. Outside, if $x^2 > \sqrt{3}(T-t)$,

$$u(t,x) = x^4 + 3(T-t)^2$$

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