13. Hamilton-Jacobi equations, Viscosity solutions

We want to consider the variational problem

$$u(t,x) = \sup_{\substack{x(\cdot)\\ x(t)=x}} \Big[\phi(x(T)) - \int_t^T h((x'(s))ds\Big]$$

with a convex h and show that for $t \le T$, $u(t,x)$ satisfies the Hamilton-Jacobi equation

$$u_t + g(\nabla u) = 0$$

with boundary condition $u(T,x) = \phi(x)$, where g is the conjugate function

$$g(x) = \sup_y [< x, y > - h(y)]$$

If h is convex the infimum

$$\inf_{\substack{x(t)=x\\ x(T)=y}} \int_t^T h(x'(s))ds = (T-t)h(\frac{y-x}{T-t})$$

so that

$$u(t,x) = \sup_y [\phi(y) - (T-t)h(\frac{y-x}{T-t})]$$

It is easy to see that

$$u(t-\delta, x) = \sup_y [u(t,y) - \delta h(\frac{y-x}{\delta})]$$

or

$$\begin{aligned} u(t-\delta,x) - u(t,x) &= \sup_y [u(t,y) - u(t,x) - \delta h(\frac{y-x}{\delta})] \\ &= \sup_z [u(t, x+\delta z) - u(t,x) - \delta h(z)] \end{aligned}$$

If we divide by δ and let $\delta \to 0$,

$$-u_t(t,x) = \sup_z [< z, \nabla u(t,x) > - h(z)] = g(\nabla u(t,x))$$

or

$$u_t + g(\nabla u) = 0$$

The question is, in what sense is the equation satisfied and is the solution uniquely determined by the equation ? The problem is that u may not be very smooth. In fact it may not even be be continuously differentiable. Let us look at some examples. Suppose $h(x) = \frac{1}{2}x^2$ and $\phi(x) = x$. Then $g(x) = \frac{1}{2}x^2$ and

$$u(t,x) = \sup_y [y - \frac{(y-x)^2}{2(T-t)}] = x + \frac{1}{2}(T-t)$$

is smooth and satisfies

$$u_t + \frac{1}{2}u_x^2 = 0$$

with $u(T, x) = x$.

But there are other spurious solutions as well. If $\phi(x) = 0$ clearly $u(t, x) \equiv 0$ is the natural solution. But there are others. For instance

$$u(t,x) = \begin{cases} 0 & \text{if} \quad -\infty < x \le -\frac{c(T-t)}{2} \\ cx + \frac{c^2(T-t)}{2} & \text{if} \quad -\frac{c(T-t)}{2} \le x \le 0 \\ -cx + \frac{c^2(T-t)}{2} & \text{if} \quad 0 \le x \le \frac{c(T-t)}{2} \\ 0 & \text{if} \quad \frac{c(T-t)t}{2} \le x < \infty \end{cases}$$

is a solution for every $c > 0$. The spurious solution, while it is not continuously differentiable, is continuous and piecewise smooth.

Theorem. Let $u(t, x)$ be a smooth solution of

$$u_t + g(\nabla u) = 0$$

in $[0, T] \times R^d$. Then

$$u(0, x) = \sup_y \Big[u(T, y) - Th(\frac{y - x}{T})\Big]$$

Proof: Consider

$$F(t) = u(t, x + at) - th(a)$$

for $a \in R^d$.

$$\begin{aligned} F'(t) &= u_t(t, x + at) + < a, \nabla u(t, x + at) > -h(a) \\ &= -g(\nabla u(t, x + t)) + < a, \nabla u(t, x + at) > -h(a) \\ &\le 0 \end{aligned}$$

Therefore,

$$u(0, x) = F(0) \ge F(T) = u(T, x + aT) - Th(a)$$

Since this is true for every $a \in R^d$,

$$u(0, x) \ge \sup_a [u(T, x + aT) - Th(a)] = \sup_y [u(T, y) - Th(\frac{y - x}{T})]$$

To prove the other half of the relation, note that

$$g(\nabla u(t, x)) =< p(t, x), \nabla u(t, x) > -h(p(t, x))$$

for the choice of $p(t, x) = (\nabla h)(\nabla u(t, x))$. Consider a solution of the ODE

$$x'(t) = p(t, x(t)) \; ; x(0) = x$$

and the function

$$G(t) = u(t, x(t)) - \int_0^t h(x'(s))ds$$

Then,

$$G'(t) = u_t(t, x(t)) + < p(t, x(t)), \nabla u(t, x(t)) > - h(x'(t)) = 0$$

Therefore

$$u(0, x) = u(T, x(T)) - \int_0^T h(x'(s))ds \le \sup_y [u(T, y) - T h(\frac{y - x}{T})]$$

The only regularity we needed was the continuity of $p(t, x)$. It is enough if u is C^1.

Remark. It is easy to construct examples where

$$u(t, x) = \sup_y [\phi(y) - (T - t) h(\frac{y - x}{T - t})]$$

is not C^1. Then there cannot be any C^1 solution. Consider the case where $\phi(y) = \cos y$ and $h(x) = \frac{x^2}{2}$. For $(T - t) = k$ large enough we have to consider a maximization of the form

$$u(T - k, x) = \sup_y [\cos y - \frac{(y - x)^2}{2k}]$$

The maximizing $y^*(x)$ as a function of x, will avoid the valleys of $\cos y$. It can always get to a y that is a multiple of 2π with a small cost if k is large. On the other hand $y^*(x)$ cannot be too far away from x. It must jump from one peak of $\cos y$ to the next one as x varies. These jumps introduce discontinuities in u_x.

$$u_x = -[\sin y^*(x)\ y^*_x(x) + \frac{y^*(x) - x}{k}(y^*_x(x) - 1)]$$

From the equation

$$\sin y^*(x) + \frac{(y^*(x) - x)}{k} = 0$$

Therefore

$$u_x = \frac{y^*(x) - x}{k}$$

and the discontinuities of y^* show up in u_x.

The question then is how to characterize the real solution?

Viscosity Solutions.

Given a bounded continuous function $u(t,x)$ on $[0,T]\times R^d$ we say that it is a **viscosity solution** of

$$u_t + g(\nabla u) = 0$$

if at any $(t_0,x_0)\in(0,T)\times R^d$, the following hold:

1. If v is any smooth function such that $v-u$ has a local maximum at (t_0,x_0) then

$$v_t(_0,x_0) + g(\nabla v(t_0,x_0)) \le 0$$

2. If v is any smooth function such that $v-u$ has a local minimum at (t_0,x_0) then

$$v_t(t_0,x_0) + g(\nabla v(t_0,x_0)) \ge 0.$$

Theorem. The Lax-Oleinik solution

$$u(t,x) = \sup_y [\phi(y) - (T-t)h(\frac{y-x}{T-t})]$$

is the unique viscosity solution of

$$u_t + g(\nabla u) = 0$$

satisfying the boundary condition

$$u(T,x) = \phi(x).$$

Proof: First let us prove that if a viscosity solution u is differentiable at some point $(t_0,x_0)\in(0,T)\times R^d$ then $u_t(t_0,x_0) + g(\nabla u(t_0,x_0)) = 0$ at that point.

To see this, if u is differentiable at (t_0,x_0), we will first construct C^1 function v such that $v-u$ has a strict local minimum at (t_0,x_0). Then we will approximate v by smooth functions v_ϵ in the C_1 topology. Since $v-u$ has a strict local minimium at (t_0,x_0) and $v_\epsilon\to v$, $v_\epsilon - u$ will have a local minimum near (t_0,x_0) say $(t_\epsilon,x_\epsilon)\to(t_0,x_0)$ as $\epsilon\to 0$. Since u is a viscosity solution, v_ϵ is smooth and $v_\epsilon - u$ has a minimum at (t_ϵ,x_ϵ) we have

$$v_{\epsilon,t}(t_\epsilon,x_\epsilon) + g(\nabla v_\epsilon(t_\epsilon,x_\epsilon)) \ge 0$$

Letting $\epsilon\to 0$ we get

$$v_t(t_0,x_0) + g(\nabla v(t_0,x_0)) \ge 0$$

the other half is similar so that we get

$$v_t(t_0,x_0) + g(\nabla v(t_0,x_0)) = 0$$

Lax-Oleinik solution is a viscosity solution. This requires us to establish **1.** and **2.** Let us start with **1.** Assume that $t_0 = x_0 = 0$. We have

$$v(0,0) = u(0,0) = \sup_y [u(\delta, y) - \delta h(\frac{y}{\delta})] \geq \sup_y [v(\delta, y) - \delta h(\frac{y}{\delta})]$$

Replacing y by δa, we get

$$\sup_a [v(\delta, \delta a) - v(0,0) - \delta h(a)] \leq 0$$

dividing by δ and taking limits, we get

$$\sup_a [v_t(0,0) + < a, \nabla v(0,0) > - h(a)] = v_t(0,0) + g(\nabla v(0,0)) \leq 0$$

Now to **2.**

$$v(0,0) = u(0,0) = \sup_y [u(\delta, y) - \delta h(\frac{y}{\delta})] \leq \sup_y [v(\delta, y) - \delta h(\frac{y}{\delta})]$$

Replacing y by δa, we get

$$\sup_a [v(\delta, \delta a) - v(0,0) - \delta h(a)] \geq 0$$

dividing by δ and taking limits, we get

$$\sup_a [v_t(0,0) + < a, \nabla v(0,0) > - h(a)] = v_t(0,0) + g(\nabla v(0,0)) \geq 0$$

Viscosity solution is unique.

We will not prove it. But refer to the text by Craig Evans for instance.

Some Comments on piecewie smooth solutions of the Hamilton-Jacobi equation in one dimension.

Suppose we have a continuosly differentiable curve $x = x(t)$ defined for $t_0 - \delta \leq t \leq t + \delta$ (passing through (t_0, x_0) where $x_0 = x(t_0)$) and a piecewise smooth function

$$u(t,x) = \begin{cases} u^+(t,x) & \text{for} \quad x(t) \leq x \leq x(t) + \delta, \ |t - t_0| < \delta \\ u^-(t,x) & \text{for} \quad x - \delta \leq x \leq x(t), \quad |t - t_0| < \delta \end{cases}$$

with both $u^{\pm}(t,x)$ being smooth functions (continuously differentiable) on its side of the curve satisfying the equation

$$u_t + g(u_x) = 0$$

for $x \neq x(t)$. We will assume that u is continuos across the curve, i.e.

$$u^+(t, x(t)) = u^-(t, x(t))$$

for $|t - t_0| < \delta$. The derivatives $u_x^{\pm}(t, x(t))$ do not necessarily match. We want to investigate the realtionship between $u_x^+(t_0, x_0)$ and $u_x^-(t_0, x_0)$

Remark 1.

Since $u^+(t, x(t)) \equiv u^-(t, x(t))$ it follows that

$$\frac{d}{dt}u^+(t, x(t)) = \frac{d}{dt}u^-(t, x(t))$$

at $t = t_0$. In other words

$$u_t^+(t_0, x(t_0)) + x'(0)u_x^+(t_0, x(t_0)) = u_t^-(t_0, x(t_0)) + x'(0)u_x^-(t_0, x(t_0))$$

or

$$-g(u_x^+(t_0, x(t_0))) + x'(t_0)u_x^+(t_0, x(t_0)) = -g(u_x^-(t_0, x(t_0))) + x'(t_0)u_x^-(t_0, x(t_0))$$

or

$$g(u_x^+(t_0, x(t_0))) - g(u_x^-(t_0, x(t_0))) = x'(t_0)[u_x^+(t_0, x(t_0)) - u_x^-(t_0, x(t_0))]$$

Remark 2.

We hope to have the variational formula

$$u(t_0, x) = \sup_y \, [\phi(y) - \frac{1}{T - t_0}h(\frac{y - x}{T - t_0})]$$

and if the supremum for $x = x_0$ is attained at some point y^*.

$$u(t_0, x_0) = \phi(y^*) - \frac{1}{T - t_0}h(\frac{y^* - x_0}{T - t_0})$$

If $x \neq x_0$, from the variational formula we get

$$u(t_0, x) \geq \phi(y^*) - \frac{1}{T - t_0}h(\frac{y^* - x}{T - t_0}) = k(x)$$

for some smooth $k(x)$ so that

$$u(t_0, x) - u(t_0, x_0) \geq k(x) - k(x_0)$$

This tells us that

$$u_x^+(t_0, x_0) \geq k'(x_0) \geq u_x^-(t_0, x_0)$$

or $u_x^+(t_0, x_0) \geq u_x^-(t_0, x_0)$.

Remark 3.

A piecewise smooth solution u is a viscosity solution at (t_0, x_0) if and only if

$$u_x^+(t_0, x_0) \geq u_x^-(t_0, x_0)$$

To see the sufficiency, suppose, for some smooth v, $v - u$ has a local minimum at (t_0, x_0). Then $v(t_0, x) - u^+(t_0, x) \geq v(t_0, x_0) - u^+(t_0, x_0)$ for $x \geq x_0$. In particular

$$v_x(t_0, x_0) \geq u_x^+(t_0, x_0)$$

and in a similar fashion

$$v_x(t_0, x_0) \leq u_x^-(t_0, x_0)$$

implying that $v_x(t_0, x_0) = u_x^+(t_0, x_0) = u_x^-(t_0, x_0)$. From the equation it follows that $u_t^\pm(t_0, x_0) = -g(u_x^\pm(t_0, x_0)) = -g(v_x(t_0, x_0))$. From the fact that $v \geq u^\pm$ and the first derivatives of $u^\pm$ match at (t_0, x_0) we conclude that in fact

$$v_t(t_0, x_0) + g(v_x(t_0, x_0)) = 0$$

On the other hand, if $v - u$ has a local maximum at (t_0, x_0), we can conclude only that

$$u_x^+(t_0, x_0) \geq v_x(t_0, x_0) \geq u_x^-(t_0, x_0)$$

and

$$v_t(t_0, x_0) + x'(0)v_x(t_0, x_0) = u_t^+(t_0, x_0) + x'(0)u_x^+(t_0, x_0) = u_t^-(t_0, x_0) + x'(0)u_x^-(t_0, x_0)$$

Hence

$$\begin{aligned} v_t(t_0, x_0) + g(v_x(t_0, x_0)) &= g(v_x(t_0, x_0)) + u_t^+(t_0, x_0) + x'(0)u_x^+(t_0, x_0) - x'(0)v_x(t_0, x_0) \\ &= g(v_x(t_0, x_0)) - g(u_x^+(t_0, x_0)) + x'(0)[u_x^+(t_0, x_0) - v_x(t_0, x_0)] \end{aligned}$$

We know from Remark 1, that the convex function

$$g(c) - g(u^+(t_0, x_0)) + x'(0)[u_x^+(t_0, x_0)) - c]$$

vanishes when $c = u^\pm(t_0, x_0)$ and is therefore nonpositive when $c = v_x(t_0, x_0)$.

Now we turn to necessity. Suppose $u_x^+(t_0, x_0) < u_x^-(t_0, x_0)$. Pick c such that

$$u_x^+(t_0, x_0) < c < u_x^-(t_0, x_0)$$

Consider the function

$$v(t, x) = u^\pm(t, x(t)) + c(x - x(t)).$$

The function $v-u$ to has a local minimum at (t_0,x_0). At (t_0,x_0),

$$v_t+g(v_x)=u_t^{\pm}+x'(0)u_x^{\pm}-cx'(0)+g(c)=g(c)-g(u_x^{\pm})+x'(0)[u_x^{\pm}-c]<0$$

providing us with a contradiction.

More general variational problems.

Suppose we have the optimization problem

$$u(t,x)=\sup_{x(\cdot):x(t)=x}[\phi(x(T))-\int_t^T h(s,x(s),x'(s))ds]$$

where $h(s,x,y)$ is a convex function of y for each s,x. The conjugate convex function $g(s,x,p)$ is defined as

$$g(s,x,p)=\sup_y[<p,y>-h(s,x,y)]$$

and the Hamilton-Jacobi equation takes the form

$$u_t(t,x)+g(t,x,\nabla u(t,x))=0$$

with the boundary condition $u(T,x)=\phi(x)$. The notion of viscosity solution and the results are analogous. One essential difference is that the minimizing paths are no longer staright lines, and the problem cannot be reduced to a simpler variational form.

Remark. We can think of the problem as a control problem with

$$dx(t)=u(t)dt$$

and a payoff function

$$f(x(T))-\int_s^T h(t,x(t),u(t))dt$$

Her there is no noise. But we could add noise and then we get the type of equations we discussed earlier. If the noise is degnerate the solutions may turnout to be nonsmooth. The notion of viscosity solution still applies. If the control parameter is only in the drift term, and not in the noise term, the second order term is linear and the solutions are generally smooth in this case. There is then the possibility of recovering the true solution in the first order case by adding a small noise and then letting it go to zero. We recover $u(t,x)=\lim_{\epsilon\to 0}u^\epsilon(t,x)$ where

$$u_t^\epsilon(t,x)+g(t,x,\nabla u^\epsilon(t,x))+\frac{1}{2}\Delta u^\epsilon(t,x)=0$$

This is the origin of the term viscosity solution.