

## 11. Boundary Conditions.

There are various contexts where a function satisfies a partial differential equation only in part of the space and some boundary conditions are used to determine the function. The equations we considered before of the form

$$u_t(t, x) + \sum_j b_j(t, x)u_{x_j}(t, x) + \frac{1}{2} \sum_{i,j} a_{i,j}(t, x)u_{x_i, x_j}(t, x) = 0$$

for  $x \in R^d, t \leq T$  is an example. The boundary is  $t = T$  and the value of  $u(T, x) = f(x)$  is provided to be able to determine  $u$ .

The Dirichlet problem is one involving only spatial variables and seeks a function  $u(x)$  in a bounded set  $G \subset R^d$ , that satisfies

$$\Delta u = \sum_i u_{x_i x_i} = 0$$

with  $u(y) = f(y)$  specified on the boundary  $\partial G$  of  $G$ . If  $d = 2$ , and  $G$  is the circle  $\{(x, y) : x^2 + y^2 < 1\}$  with  $\partial G = \{(x, y) : x^2 + y^2 = 1\}$ , we are seeking a harmonic function with boundary values  $f$  on the circle. The answer is explicitly given by means the Poisson kernel and in polar coordinates

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)f(\varphi)}{1 - 2r \cos(\theta - \varphi) + r^2} d\varphi$$

where the point on the circle is represented as  $(1, \varphi)$  in polar coordinates and  $x = (r, \theta)$ . While it is not possible to write an explicit answer in general, if the boundary  $\partial G$  is smooth, the Dirichlet problem has a nice answer and

$$(1) \quad u(x) = \int_{\partial G} K(x, z)f(z)d\sigma(z)$$

where  $d\sigma$  is the  $d - 1$  dimensional surface area and  $K(x, z)$  is the analog of the Poisson kernel. The Poisson kernel has the following additional properties:

1.  $K(x, z) \geq 0$
2.  $\int K(x, z)d\sigma(z) = 1$  for all  $x \in G$ .
3. For each  $z$ ,  $\Delta K(\cdot, z) = 0$  i.e.  $K$  is a harmonic function of  $x$  for each  $z$ .
4. For any continuous function  $f$  on  $\partial G$

$$\lim_{\substack{x \rightarrow y \\ x \in G \\ y \in \partial G}} \int_{\partial G} K(x, z)f(z)d\sigma(z) = f(y)$$

It is clear that the properties of  $K$  imply that  $u$  given by (1) solves the Dirichlet problem. An easy consequence is the maximum principle

A harmonic function achieves its maximum and minimum at the boundary. In particular if  $f$  is nonnegative so is  $u$ . If  $f$  is bounded by  $C$  so is  $u$ . The value  $u(x)$  is an average of the values of  $f(y)$  on the boundary,  $K(x, z)$  being the weight that determines the average for  $x$ .

There is a probabilistic interpretation for this averaging. Let us start a  $d$ -dimensional Brownian motion  $\beta(\cdot)$  from  $x \in G$ . We saw that the one dimensional Brownian motion will eventually reach any value. In particular the  $d$ -dimensional Brownian motion will exit  $G$  sooner or later. Its exit point will be some random point on the boundary  $\partial G$ . let us call the time of first exit  $\tau$  and the exit place  $\beta(\tau) \in \partial G$ . Then the solution to the Dirichlet problem is

$$u(x) = E \left[ f(\beta(\tau)) \mid \beta(0) = x \right]$$

There is a quick way of seeing this connection. If  $x \in G$  let us pick a small ball of radius  $r$  such that  $B(x, r) \subset G$ . then the Brownian path has to exit from the small ball at some time  $\tau_1$  at the point  $\beta(\tau_1)$  and then start afresh from that point and make its way to the boundary of  $G$ . So we can write

$$\begin{aligned} u(x) &= E \left[ E \left[ f(\beta(\tau)) \mid \beta(0) = \beta(\tau_1) \right] \mid \beta(0) = x \right] \\ &= E \left[ u(\beta(\tau_1)) \mid \beta(0) = x \right] \end{aligned}$$

But Brownian motion is invariant under rotations. So the exit place on the Boundary of  $B(x, r)$  is uniform on the surface of the sphere. In other words  $u$  has the mean value property and is therefore harmonic and satisfies  $\Delta u = 0$ . If  $x$  is close to the boundary it will exit quickly and so the exit place  $\beta(\tau)$  cannot be far from  $x$ . This guarantees that  $u(x) \rightarrow f(y)$  as  $x \rightarrow y$ .

There is different argument that is natural as well. Let us suppose that  $u(x)$  is a solution of  $\Delta u = 0$  with  $u = f$  on the boundary. In fact let us suppose that  $u(x)$  is smooth and defined everywhere but satisfies  $u = f$  on  $\partial G$  and  $\Delta u = 0$  in  $G$ . By Itô's formula

$$u(\beta(t)) - u(x) = \int_0^t \nabla u(\beta(s)) \cdot d\beta(s) + \frac{1}{2} \int_0^t (\Delta u)(\beta(s)) ds$$

Since this is valid for each path we can repalce  $t$  by  $\tau$  that is random.

$$u(\beta(\tau)) - u(x) = \int_0^\tau \nabla u(\beta(s)) \cdot d\beta(s) + \frac{1}{2} \int_0^\tau (\Delta u)(\beta(s)) ds$$

Since  $\beta(t) \in G$  for  $t \leq \tau$  it follows that  $(\Delta u)(\beta(t)) = 0$  for  $t \leq \tau$ . therefore

$$u(\beta(\tau)) - u(x) = \int_0^\tau \nabla u(\beta(s)) \cdot d\beta(s)$$

Because  $\tau$  is a stopping time (one can argue with a little bit of effort) that

$$E \left[ \int_0^\tau \nabla u(\beta(s)) \cdot d\beta(s) \mid \beta(0) = x \right] = 0$$

It now follows that

$$u(x) = E \left[ u(\beta(\tau)) \mid \beta(0) = x \right] = E \left[ f(\beta(\tau)) \mid \beta(0) = x \right]$$

The averaging is then just the expected value of  $f(\beta(\tau))$ , the value of  $f$  at the random exit place.

**Warning.** One has to be a bit careful. If we look at one dimensional Brownian motion starting from 0, and  $\tau$  is the first hitting time of  $x = 1$ , then

$$\beta(\tau) - \beta(0) = 1 - 0 = \int_0^\tau d\beta(s)$$

does not have mean zero although  $\tau$  is a stopping time. This is the gambler's paradox!. If the game is fair and there is probability one that the gambler will be ahead at some point, why not wait till one is ahead and then quit? The answer lies in the fact that before one is sure to reach 1 there is a chance that  $\beta(t)$  will see very large negative values. The strategy works only if there is an infinite credit line. If one wants to stop at a stopping time, to be absolutely sure one has to use a **bounded** stopping time. If  $\tau$  is not bounded then  $\tau_n = \min(\tau, n)$  is a bounded stopping time. In both cases we can conclude respectively that

$$u(x) = E[u(\beta(\tau_n)) \mid \beta(0) = x] ; E[\beta(\tau_n)] = 0$$

In the first case we can let  $n \rightarrow \infty$  easily because  $u$  is bounded. In the second case, we cannot because  $\beta(\tau_n)$  can be arbitrarily large negative. We can only be sure that it belongs to  $(-\infty, 1]$ . Typically  $\beta(\tau_n)$  can be 1 with probability  $(1 - \frac{1}{n})$  and  $-n$  with probability  $\frac{1}{n}$ . Each expectation is 0. But in the limit the expectation jumps to 1.

More generally we can replace  $\Delta u = 0$  by

$$(2) \quad \frac{1}{2} \sum_{i,j} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_j b_i(x) \frac{\partial u}{\partial x_i} = 0 \quad \text{for } x \in G$$

with

$$(3) \quad \lim_{x \rightarrow y} u(x) = f(y) \quad \text{for } x \in \partial G$$

If  $G$  is a bounded set with smooth boundary and the coefficients are smooth and bounded this has a solution provided the symmetric  $\{a_{i,j}(x)\}$  is uniformly positive definite, i.e.

$$(4) \quad \sum_{i,j} a_{i,j}(x) \xi_i \xi_j \geq c \sum_j \xi_j^2$$

for some  $c > 0$  and all  $\xi \in R^d$  and  $x \in G$ . The theory is no different than the case of the Laplacian  $\Delta$ . One takes the solution of the SDE

$$dx_i(t) = b_i(x(t))dt + \sum_j \sigma_{i,j}(x(t))d\beta_j(x(t))$$

and defines  $\tau = \inf[t : x(t) \notin G]$  the exit time. Then

$$(5) \quad u(x) = E \left[ f(x(\tau)) \mid x(0) = x \right]$$

is the solution.

We can try to solve more general equations of the form

$$(6) \quad \frac{1}{2} \sum_{i,j} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_j b_i(x) \frac{\partial u}{\partial x_i} - V(x)u(x) + g(x) = 0 \quad \text{for } x \in G$$

with the boundary condition (3). The solution exist provided  $V(x) \geq 0$  and is given by

$$(7) \quad u(x) = E \left[ f(x(\tau)) \exp \left[ - \int_0^\tau V(x(s))ds \right] + \int_0^\tau g(x(s)) \exp \left[ - \int_0^s V(x(t))dt \right] ds \mid x(0) = x \right]$$

which is the expected value of the sum of two discounted payoffs, a terminal payoff of  $f(x(\tau))$  that depends on the exit place and a running payoff  $g(x(s))$  depending on the location at time  $s$ , the discount factor  $\exp[-\int_0^s V(x(t))dt]$  depending on the past history.

### Examples:

1\*. If  $u$  satisfies (2) with the boundary condition (3), use Itô's formula to prove (5).

The nondegeneracy condition (4), among other things will guarantee that  $\tau < \infty$  and in fact can even get bounds on  $E[\tau \mid x(0) = x]$ . But if the  $\{a_{i,j}(x)\}$  begin to degenerate near the boundary, the boundary may never be reached. In such case the Dirichlet problem may not have a good solution. For instance in one dimension the geometric Brownian motion

$$dx(t) = ax(t)dt + \sigma x(t)d\beta(t) ; x(0) = x > 0$$

will never reach 0, as can be seen by explicit calculation.

It is important to able to prove it without exact calculation. Consider the function  $x^{-c}$  for some  $c > 0$ . Then

$$\left[ax\frac{d}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2}{dx^2}\right]x^{-c} = \left[-ac + \frac{c(c+1)\sigma^2}{2}\right]x^{-c} = kx^{-c}$$

We can pick  $c > 0$  such that  $k > 0$ . Itô's formula now tells us that

$$x(t)^{-c}e^{-kt} - x^{-c} = \int_0^t g(x(s))d\beta(s)$$

without any  $dt$  term. If we do not wait to hit 0 but stop when we hit  $\epsilon > 0$ , for the hitting time  $\tau_\epsilon$  of  $\epsilon$  we get

$$E[\epsilon^{-c}e^{-k\tau_\epsilon}|x(0) = x] = x^{-c}$$

or

$$E[e^{-k\tau_\epsilon}|x(0) = x] = \left(\frac{\epsilon}{x}\right)^c$$

If we now let  $\epsilon \rightarrow 0$  we see that  $\tau_\epsilon \rightarrow \infty$ .

In general it is possible to estimate solutions of PDE's with out actually solving them by the use of maximum principle. Suppose we want to esitmate the solution of equation (6) with boundary condition (3). If we find a  $w$  such that

$$(8) \quad \frac{1}{2} \sum_{i,j} a_{i,j}(x) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_j b_i(x) \frac{\partial w}{\partial x_i} - V(x)w(x) + g(x) = -\hat{g}(x) \leq 0 \quad \text{for } x \in G$$

with  $\lim_{x \rightarrow y} w(x) = \hat{f}(y) \geq f(y)$  on the boundary, then  $w$  satis fies the equation with  $g + \hat{g}$  and  $\hat{f}$  which are larger than  $g$  and  $f$ . By the representation formula (7) it is clear that  $w$  provides an upper bound for  $u$ .

2. For instance suppose we want to estimate the solution  $u$  of

$$\frac{1}{2} \sum_{i,j} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + 1 = 0$$

on a ball of radius 1 with  $u = 0$  on the boundary. Assuming

$$C \sum_i \xi_i^2 \geq \sum_{i,j} a_{i,j}(x) \xi_i \xi_j \geq c \sum_i \xi_i^2$$

the function  $v(x) = \frac{(1-r^2)}{cd}$  is concave and satisfies

$$\frac{c}{2} \Delta v + 1 = 0$$

Therefore

$$\frac{1}{2} \sum_{i,j} a_{i,j}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + 1 \leq \frac{c}{2} \Delta v + 1 = 0$$

and  $u(x) \leq \frac{1-r^2}{cd}$ . By a similar argument  $u(x) \geq \frac{1-r^2}{Cd}$ .

3\*. Let us suppose that  $|b(x)| \leq Cx$  and  $|\sigma(x)| \leq Cx$  for some  $C < \infty$  and  $x > 0$ . Then the solution of

$$dx(t) = b(x(t))dt + \sigma(x(t))d\beta(t) ; \quad x(0) = x > 0$$

will never reach 0, just like the geometric Brownian Motion.

Let us look at the general formula (7) for the solution of (6). Suppose  $G$  is an unbounded set and or the coefficient matrix  $\{a_{i,j}(x)\}$  is degenerate. The worst thing to happen might be that  $\tau$  may not be finite or even if it is finite may have a tail that is too fat. large values are not too rare. So long as it is finite and  $V \geq 0$ , there is no problem. the integral is defined and the problem is solvable. If  $\tau$  can be infinite with some probability, then if  $V \geq c > 0$ , the discount kills all terms with  $\tau = \infty$  and we do not have to worry about it. The trouble is only when  $\tau$  can be infinite and  $V$  is not uniformly positive. The boundary condition needs to be specified only on that part of the boundary that can be reached. If there is explicit time dependence we can just treat as one extra coordinate. For instance

$$u_t = u_{xx}$$

is no different from

$$u_{yy} - u_x = 0$$

The SDE are trivial  $x(t) = x - t$  and  $y(t) = y + \beta(t)$ . The solution is  $(x - t, y + \beta(t))$ . If we want to solve it in a square, since time moves to the left, the right boundary is never reached. Either the path exits from the top or bottom boundary or from the left boundary. In any case  $\tau$  cannot exceed the width of the square. So we can specify  $f$  on the three sides that matter and the solution exists in this case no matter what  $V$  and  $g$  are. The time dependent parabolic case is not conceptually different from equations that do not explicitly depend on time.

The type of boundary conditions that we have considered are generally called Dirichlet type conditions. In terms of the solution  $x(t)$  of the SDE, the game is over when the boundary is reached and the boundary conditions determine the payoff that depends on when and where the boundary is reached. Another type of boundary condition is called the Neumann type boundary condition. This is best explained in the random walk model. Imagine a fair gambling game or a random walk, where the current assets  $S_n$  either increase or decreases by 1 so that  $S_{n+1} = S_n \pm 1$  with probability  $\frac{1}{2}$ . Since a person with no assets cannot play the game, if one starts initially with  $S_0 = x$ , the game ends whenever  $S_k = 0$  for some  $k$ . Unless there is a dutch uncle who provides a dollar whenever  $S_n = 0$ , moving the current assets to 1. In other words if it is the lost dollar that has to be paid, the uncle will pay it instead. Now the game can go on for ever. Eventually a run of good luck might make the assets grow providing a period of respite for the uncle. But only to be visited again by misfortune and rescue by the uncle. The way to model is by two processes, the random walk  $S_n$  and the current assets  $A_n$

$$A_n = S_n + C_n$$

where  $C_n$  is increasing and is the samllest amount needed to maintain  $A_n \geq 0$ . If we scale it and pass to the continuous limit

$$x(t) = \beta(t) + C(t)$$

where  $C(t)$  is an increasing function. It is used to keep  $x(t) \geq 0$  and will increase only when  $x(t) = 0$ . In fact

$$\begin{aligned} C(t) &= - \inf_{0 \leq s \leq t} \beta(s) \\ x(t) &= \beta(t) - \inf_{0 \leq s \leq t} \beta(s) \end{aligned}$$

It turns out that

$$E[f(x(t))|x(0) = x]$$

will be now a solution of

$$u_t = \frac{1}{2}u_{xx} ; \quad u_x(t, 0) = 0 ; \quad u(0, x) = f(x)$$

and this is the Neumann boundary condition. Itô's formula is similar,

$$du(t, x(t)) = u_t dt + u_x dx + \frac{1}{2}u_{xx} dt = [u_t + \frac{1}{2}u_{xx}(t, x(t))]dt + u_x(t, 0)dC(t) + u_x(t, x(t))d\beta(t)$$

**Example:** 4. Show that the solution is explicitly given by

$$u(t, x) = \int_0^\infty f(y) \frac{1}{\sqrt{2\pi t}} [e^{\frac{-(y-x)^2}{2t}} + e^{\frac{-(y+x)^2}{2t}}] dy$$