10. Connections between PDE's and Diffusions.

Let us now take a step back and look at the types of differential equations that we have seen.

1. First order linear differential equations of the form

$$b(x) \cdot \nabla u = 0$$

for $x \in G \subset R^d$, with $u(y) = f(y)$ for $y \in \Gamma \subset \partial G$. The solution was given by solving the ODEs

$$x'(t) = b(x(t))$$

and assuming that each trajectory meets Γ exactly once the value of u at any point on the trajectory was given by the value $f(y)$ of f at the point y on the boundary where the trajectory meets Γ. The rationale is that

$$\frac{d}{dt} u(x(t)) = x'(t) \cdot \nabla u(x(t)) = b(x(t)) \cdot \nabla u(x(t)) = 0$$

and so u must be constant on characteristics i.e. solutions of $x'(t) = b(x(t))$.

2. A special case where one is always sure that every trajectory meets Γ exactly once is when $b_1(x) \equiv 1$ and $G = \{x : x_1 \leq T\}$ and $\Gamma = \{x : x_1 = T\}$. Now $x_1(t) = x_1(0) + t$ and if $x_1 < T$, meets $x_1 = T$ exactly once at $t = T - x_1(0)$. We prefer to call $x_1 = t$ and $x = (x_2, \cdots, x_d)$. Then the equation becomes

$$\frac{\partial u}{\partial t} + < b(t, x), \nabla_x u(t, x) >= 0$$

with $u(T, y) = f(y)$. To find the solution $u(s, x)$ for some $s < T$ we solve $x'(t) = b(t, x(t))$ with $x(s) = x$ and Γ is met at $x(T)$ so that $u(s, x) = f(x(T))$. Of course the valus of $x(T)$ depends on s and x.

3. The heat equation in R.

$$\frac{\partial u}{\partial t} + b(t, x) u_x(t, x) + \frac{a(t, x)}{2} u_{xx}(t, x) = 0$$

with $u(T, y) = f(y)$. Now there is no single trajectory emanating from (s, x). Instead there is a whole bunch of them depending on a random Brownian path $\beta(\cdot)$. The random characteristic corresponding to a random Brownian path $\beta(\cdot)$ is obtained by solving

$$dx(t) = b(t, x(t))dt + \sqrt{a(t, x(t))} d\beta(t) \; ; x(s) = x$$

The random trajectory $(t, x(t))$ meets Γ, which is $t = T$, at the point $x(T)$ which is again random. The value $u(s, x)$ of the solution is again similar to the first order case in that it is the average value of $f(x(T))$ given by

$$u(s, x) = E[f(x(T))|x(s) = x]$$

3. Now we can generalize to the higher dimensional case, where the equation is

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} b_i(t, x)\frac{\partial u}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(t, x)\frac{\partial^2 u}{\partial x_i \partial x_j} = 0$$

with $u(T, y) = f(y)$ for $y \in R^d$. The condition that $a(s, x) \geq 0$ is replaced by the condition that the matrix $\{a_{i,j}(s, x)\}$ be symmetric and positive semidefinite for each (s, x). The role of $\sqrt{a(s, x)}$ is now played by a matrix $\sigma(s, x)$ with the property $\sigma(s, x)\sigma^*(s, x) = a(s, x)$ for each (s, x). Here $*$ represents taking tanspose, and the relation really is

$$a_{i,j}(s, x) = \sum_k \sigma_{i,k}(s, x)\sigma_{j,k}(s, x)$$

for all i, j, s and x. The interpretation is that if ξ is a standard d-dimensional Gaussian random vector with mean 0 and covariance $I = \{\delta_{i,j}\}$, then $\eta = \sigma(s, x)\xi$ given by $\eta_i = \sum_{j=1}^{d} \sigma_{i,j}(s, x)\xi_j$ will be a d-dimensional Gaussian vector with mean 0 and covariance $a_{i,j}(s, x)$. The equation we need to solve is the system

$$dx_i(t) = b_i(t, x(t))dt + \sum_{j=1}^{d} \sigma_{i,j}(t, x(t))d\beta_j(t) \;; x_i(s) = x_i$$

where $\beta(\cdot) = \{\beta_i(\cdot)\}$ are d independent Brownian motions and $x = (x_1, \cdots, x_d)$ is the initial condition at time s. There is not much difference in proving existence and uniqueness under Lipschitz condition between the one dimensional and multidimensional case. The value $f(x(T))$ of f at the random point where Γ is met is averaged over all $\beta(\cdot)$ to get

$$u(s, x) = E[f(x(T))|x(s) = x]$$

Itô's formula is still nearly the same with one modification. $(d\beta_i)^2 = dt$ as before. But the independence makes $d\beta_i d\beta_j = 0$ for $i \neq j$.

For instance

$$\begin{aligned}
&du(\beta_1(t), \cdots, d\beta_d(t)) \\
&\qquad = \sum_j u_j(\beta_1(t), \cdots, \beta_d(t))d\beta_j(t) + \frac{1}{2}\sum u_{i,j}(\beta_1(t), \cdots, \beta_d(t))d\beta_i(t)d\beta_j(t) \\
&\qquad = \sum_j u_j(\beta_1(t), \cdots, \beta_d(t))d\beta_j(t) + \frac{1}{2}\sum_i u_{i,i}(\beta_1(t), \cdots, \beta_d(t))dt \\
&\qquad = \sum_j u_j(\beta_1(t), \cdots, \beta_d(t))d\beta_j(t) + \frac{1}{2}(\Delta u)(\beta_1(t), \cdots, \beta_d(t))dt
\end{aligned}$$

The random solution $x(t)$ at time t, starting from x at time s will have a probability distribution computed from the Brownian motion that supplies all the randomness in the model. This distribution $\mu_{s,x,t}(dy)$ is denoted by $p(s,x,t,dy)$ and is the transition probability of the diffusion process that corresponds to the coefficients $b(s,x)$ and $a(s,x)$. If $a(s,x)$ is strictly positive definite, i.e. of full rank, then $p(s,x,t,dy)$ will have a density $p(s,x,t,y)$ which is called the fundamental solution. It will satisfy, under some mild regularity conditions the PDE

$$\frac{\partial p}{\partial s} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial u}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(t,x)\frac{\partial^2 p}{\partial x_i \partial x_j} = 0$$

for every (t,y) so long as $s < t$. Therefore any integral

$$u(s,x) = E[f(x(T)|x(s) = x] = \int_{R^d} f(y)p(s,x,T,y)dy$$

will automatically be a solution of

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial u}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} = 0$$

with $u(T,y) = f(y)$. One essential difference between the first order case and the second order case is that the random trajectories go only forward in time. In the first order equation there is no qualitative difference between b and $-b$. But now because of the positive definiteness condition there is a world of difference between a and $-a$.

4. We can unify to some extent equations of the form

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{d} b_i(t,x)\frac{\partial u}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j}+V(t,x)u(t,x)+g(t,x)=0$$

In addition to $x=(x_1,\cdots,x_d)$ let us introduce two extra coordinates x_{d+1} and x_{d+2}. They will evolve by

$$dx_{d+1}=V(t,x)x_{d+1}dt$$

and

$$dx_{d+2}=g(t,x)dt$$

We now need to look at the equation

$$\frac{\partial w}{\partial t}+\sum_{i=1}^{d} b_i(t,x)\frac{\partial w}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(t,x)\frac{\partial^2 w}{\partial x_i \partial x_j}+V(t,x)x_{d+1}\frac{\partial w}{\partial x_{d+1}}+g(t,x)x_{d+1}\frac{\partial w}{\partial x_{d+2}}=0$$

If we look for a solution of the form

$$w(t,x,x_{d+1},x_{d+2})=u(t,x)x_{d+1}+x_{d+2}$$

Then substituting w in the equation above we get

$$x_{d+1}\left[\frac{\partial u}{\partial t}+\sum_{i=1}^{d} b_i(t,x)\frac{\partial u}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j}+V(t,x)u(t,x)+g(t,x)\right]=0$$

We can recover $u(s,x)$ as $w(s,x,1,0)$. $f(y,y_{d+1},y_{d+2})=y_{d+1}f(y)+y_{d+2}$. Finally we conclude that

$$u(s,x)=E\big[x_{d+1}(T)f(x(T)))+x_{d+2}(T)|x(s)=x,x_{d+1}(s)=1,x_{d+2}(s)=0\big]$$

Note that

$$x_{d+1}(T)=\exp[\int_s^T V(t,(x(t))dt]$$

and

$$x_{d+2}(T)=\int_s^T g(t,x(t))\exp[\int_s^t V(z,x(z))dz]dt$$

So we finally get the Feynman-Kac formula

$$u(T,x)=$$
$$E\left[\exp[\int_s^T V(z,x(z))dz]f(x(T))+\int_s^T \exp[\int_s^t V(z,x(z))dz]g(t,x(t))dt|x(s)=x\right]$$

5. Seperation of variables. If $d = d_1 + d_2$ and the operator

$$L_t = \sum_{i=1}^{d} b_i(t,x)\frac{\partial u}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j}$$

is the sum of two operators

$$L_t = L_t^1 + L_t^2$$

with $x^1 = (x_1, \cdots, x_{d_1})$

$$L_t^1 = \sum_{i=1}^{d_1} b_i(t,x^1)\frac{\partial u}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d_1} a_{i,j}(t,x^1)\frac{\partial^2 u}{\partial x_i \partial x_j}$$

$x^2 = (x_{d_1+1}, \cdots, x_{d_2})$, and

$$L_t^2 = \sum_{i=d_1+1}^{d_2} b_i(t,x^2)\frac{\partial u}{\partial x_i} + \frac{1}{2}\sum_{i,j=d_1+1}^{d_2} a_{i,j}(t,x^2)\frac{\partial^2 u}{\partial x_i \partial x_j}$$

The fundamental solution factors into a product

$$p(s,x,t,y) = p_1(s,x^1,t,y^1)p_2(s,x^2,t,y^2)$$

and we have two seperate problems in d_1 and d_2 dimensions. In terms of the solution $x(t)$ of the SDE it is the solution $x^1(t), x^2(t)$ of two sets of equations one involving the first d_1 Brownian motions and the other the second set. Thr random processes $x^1(\cdot)$ and $x^2(\cdot)$ are statistically independent.

Examples:

1. For instance

$$u_t + \frac{1}{2}\Delta u = 0$$

decomposes into d one-dimensional problems of the form

$$u_t + \frac{1}{2}u_{x_i x_i} = 0$$

giving the fundamental solution

$$p(s,x,t,y) = \Pi_{i=1}^{d}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(y_i - x_i)^2}{2t}}$$

2. Ornstein-Uhlenbeck process with mean reversal.

$$dx_i(t) = \sum_j a_{i,j} x_j(t) + d\beta(t) \ ; x_i(0) = x_i$$

on R^d. If we denote by A the matrix $\{a_{i,j}\}$, then

$$x(t) = e^{At}x + \int_0^t e^{A(t-s)} d\beta(s)$$

solves the SDE. The distribution of $x(t)$ is Gaussian with mean $e^{At}x$ and covariance

$$C(t) = \int_0^t e^{A(t-s)} e^{A^*(t-s)} ds = \int_0^t e^{As} e^{A^* s} ds$$

If A has eigenvalues with negative real parts, then

$$e^{At}x \to 0$$

as $t \to \infty$ and

$$C = \int_0^\infty e^{As} e^{A^* s} ds$$

exists as a positive definite matrix, and after a long time $x(t)$ forgets its starting point and has a limiting distribution with mean 0 and covariance C. If A is symmetric

$$C = (-2A)^{-1}$$

3*. What happens if the mean reversal is not about zero, but about some $b = (b_1, \cdots, b_d)$?

$$dx_i(t) = \sum_j a_{i,j}(x_j(t) - b_j) + d\beta(t) \ ; x_i(0) = x_i$$

4. Consider the geometric Brownian motion

$$dx(t) = ax(t)dt + \sqrt{\theta(t)} x(t) d\beta_1(t) \ x(0) = x$$

on R, where the volatility $\theta(t)$ is random and evolves as

$$d\theta(t) = b(\theta(t))dt + \beta_2(t) \ ; \ \theta(0) = \theta$$

on R^+. Assume β_1, β_2 are independent Brownian motions. How would you calculate

$$E\Big[[x(t)]^\alpha | x(0) = x, \theta(0) = \theta \Big]$$

First pretending that we know $\theta(t)$

$$x(t) = x \exp[\int_0^t \sqrt{\theta(s)} d\beta(s) - \frac{1}{2}\int_0^t \theta(s) ds + at]$$

Therefore

$$\begin{aligned} E[[x(t)]^\alpha] &= x^\alpha E\left[\exp[\alpha \int_0^t \sqrt{\theta(s)} d\beta(s) - \frac{\alpha}{2}\int_0^t \theta(s) ds + a\alpha t] | \theta(0) = \theta\right] \\ &= x^\alpha E\left[\exp[\frac{\alpha^2 - \alpha}{2}\int_0^t \theta(s) ds + a\alpha t] | \theta(0) = \theta\right] \\ &= x^\alpha e^{a\alpha t} F(t, \theta) \end{aligned}$$

with F given by Feynaman-Kac formula as the solution of

$$(F - K) \qquad \frac{\partial F}{\partial t} = \frac{1}{2} F_{\theta\theta} + b(\theta) F_\theta + \frac{\alpha(\alpha - 1)}{2} \theta F \ ; \ F(0, \theta) = 1.$$

5*. Can you relate directly the solution $u(t, x, \theta)$ of

$$\frac{\partial u}{\partial t} = a x u_x + \frac{\theta x^2}{2} u_{xx} + b(\theta) u_\theta + \frac{1}{2} u_{\theta\theta} \ ; \ u(0, x, \theta) = x^\alpha$$

to the solution of $(F - K)$?