1. Recurrence Relations, timesteps and differential equations.

Often the state of a system changes over time and the new state can be described as a function of the old state. If $\mathcal{S}$ is the set of states (uasually the set of real numbers) then

$$x_{new} = f(x_{old})$$

where $f : \mathcal{S} \to \mathcal{S}$ is a function or map of $\mathcal{S}$ into itself. If the system goes through successive changes under the same general circustances, then one defines successively

$$x_k = f(x_{k-1})$$

for $k \geq 1$ with x_0 as the initial state. Then the state after k steps is given by

$$x_k = f^{(k)}(x_0)$$

where $f^{(k)}$ is the k-th iterate of f, defined inductively by

$$f^{(k)}(\cdot) = f(f^{(k-1)}(\cdot))$$

Examples:

1. Suppose at each step we add a fixed amount a to the pot. Then clearly $f(x) = x + a$, and $x_k = f^{(k)}(x_0) = x_0 + ka$.

2. Compound interest. $f(x) = (1 + r)x$. Then $x_k = (1 + r)^k \ x_0$

3*. Compounding with consumption. From an initial capital of x_0 an amount of a is consumed each year and the rest invested to produce an annual rate of return of r so that assets from the start one year to the next is related by $f(x) = (x - a)(1 + r)$. What is $f^{(k)}(x)$?. How does it behave for large k? In particular when will it always remain nonnegative? Express the condition in terms of a, x_0 and r. If it turns negative find the value of $k = k(x_0, a, r)$ when it turns negative for the first time.

Timesteps. Often, each step in the iteration represents passage of time. If the time step is small, then one expects $f(x) - x$ to be small. If the timestep is h in some time units one might expect that for small values of h, $f(x) = f_h(x) = x + h \ b(x) + o(h)$ for some function $b(x)$. If we pretend that time is continuous and the state at time t is given by a function $x(t)$, then our approximation can be written as

$$x(t + h) = x(t) + h \ b(x(t)) + o(h)$$

Formally this leads to the differential equation

$$\frac{dx(t)}{dt} = b(x(t)) \ ; \qquad x(0) = x_0$$

The theory of ordinary differential equations tells us that for any function $b(x)$ satisfying the Lipschitz condition $|b(x)-b(y)| \leq C|x-y|$ and for any initial value x_0, the above ODE has a unique solution $x(t) = f(t, x_0)$ for some f. Here $f(t, \cdot)$ replaces the iterate $f^{(k)}(\cdot)$. Note that $f(s, f(t, x)) = f(t, f(s, x)) = f(t+s, x)$ for all $x, s > 0$ and $t > 0$.

Proposition: Let $b(x)$ satisfy the Lipschitz condition. Let $f_h(x)$ be any function satisfying

$$\lim_{h \to 0} \frac{1}{h} \sup_x |f_h(x) - x - h\ b(x)| = 0$$

Then for any $t > 0$ and x_0,

$$\lim_{\substack{h \to 0 \\ k \to \infty \\ kh \to t}} f_h^{(k)}(x_0) = f(t, x_0)$$

where $f(t, x)$ is the solution at time t of the ODE

$$\frac{dx(t)}{dt} = b(x(t))\ ; \qquad x(0) = x_0$$

Proof: Let us set $y_k = f_h^{(k)}(x_0), x_k = f(hk, x_0)$ and compare the difference

$$\begin{aligned} |f_h^{(k)}(x_0) - f(hk, x_0)| &= |y_k - x_k| = |f_h(y_{k-1}) - f(h, x_{k-1})| \\ &= |y_{k-1} + h\ b(y_{k-1}) + o(h) - x_{k-1} - h\ b(x_{k-1}) - o(h)| \\ &\leq (1 + Ch)|y_{k-1} - x_{k-1}| + o(h) \end{aligned}$$

We have used the Lipschitz condition $|b(y_{k-1}) - b(x_{k-1})| \leq C|y_{k-1} - x_{k-1}|$. If we denote by $\Delta_k(h) = \sup_x |f_h^{(k)}(x) - f(kh, x)|$, then

$$\Delta_k(h) \leq (1 + Ch)\Delta_{k-1}(h) + o(h)$$

Lemma: Let $u_n \geq 0$ satisfy $u_n \leq a u_{n-1} + b$ for $n \geq 1$ with some positive a and b. Then, by induction,

$$u_n \leq a^n u_0 + b(1 + a + a^2 + \cdots + a^{n-1})$$

and therefore

$$u_n \leq a^n u_0 + nb \max\{1, a^n\}$$

Since $\Delta_0(h) = 0$, it follows that

$$\Delta_k(h) \leq k\ o(h)\ (1 + Ch)^k \to 0 \quad \text{as} \quad h \to 0,\ t \to \infty \text{ and } kh \to t.$$

Examples:

4. If $f_h(x) = x + ah + o(h)$, we get

$$\frac{dx(t)}{dt} = a$$

and

$$x(t) = x_0 + at$$

5. If $f_h(x) = x(1 + rh)$ we get

$$\frac{dx(t)}{dt} = rx(t)$$

and we get

$$x(t) = x_0 e^{r\,t}$$

6*. Do the continuous time analog of 3*. $f_h(x) = (x - ah)(1 + rh) + o(h)$. Write down the ODE. Solve it. When does the solution always remain nonnegative? If it does become negative at what time $T = T(x_0, a, r)$ does it first reach 0?. Compare it to the corresponding answer in 3*.

There are higher dimensional analogs of these. We can have $\mathbf{f}_h(\mathbf{x})$ mapping $R^d \to R^d$ satisfying

$$\mathbf{f}_h(\mathbf{x}) = \mathbf{x} + h\ \mathbf{b}(\mathbf{x}) + o(h)$$

where $\mathbf{x} = (x_1, \cdots, x_d)$,

$$\mathbf{f}(\mathbf{x}) = (f_1(x_1, \cdots, x_d), \cdots, f_d(x_1, \cdots, x_d))$$

and

$$\mathbf{b}(\mathbf{x}) = (b_1(x_1, \cdots, x_d), \cdots, b_d(x_1, \cdots, x_d)).$$

We get a system of ODEs

$$\frac{dx_i(t)}{dt} = b_i(x_1(t), \cdots, x_d(t))\ ; x_i(0) = x_i \quad i = 1, \cdots, d$$

and the corresponding solutions

$$\mathbf{x}(t) = \mathbf{f}(t, \mathbf{x})$$

mapping $R^d \to R^d$. Of particular interest are the linear equations where

$$\mathbf{b}(\mathbf{x}) = \mathbf{b}\mathbf{x} = \sum_j b_{i,j} x_j$$

for some matrix $\mathbf{b} = \{b_{i,j}\}$. One can show that in this case (which is Lipschitz) the solution is given by

$$\mathbf{x}(t) = \mathbf{F}(t)\mathbf{x}$$

with the matrix $\mathbf{F}(t)$ given by

$$\mathbf{F}(t) = \exp[t\mathbf{B}] = \sum_{k=0}^{\infty} \frac{t^k}{k!}\ \mathbf{B}^k$$

Example:

7. Let $d = 2$ and the coordinates be x and y. Assume $\mathbf{f}_h(x, y) = (x + hy, y - hx)$. We can solve explicitly for $\mathbf{F}(t)$.

$$F_{11}(t) = F_{22}(t) = \cos t \quad \text{and} \quad F_{12}(t) = -F_{21}(t) = \sin t$$

We can replace the Lipschitz condition by a local Lipschitz condition and boundedness. $|b(x)| \le C$ for some C and for every A there is C_A such that

$$|b(x) - b(y)| \le C_A|x - y| \quad \text{for} \quad x, y \in [-A, A]$$

The boundedness gaurantees existence and the local Lipschitz condition the uniqueness. Actually for existence linear growth is OK, which explains why gloabl Lipschitz works for both existence and uniqueness. If we have faster than linear growth the solution can blow up in a finite time. For instance if $b(x) = x^2$ the solution of

$$\frac{dx(t)}{dt} = x^2(t)\ ;\ x(0) = 1$$

is $x(t) = (1 - t)^{-1}$ which blows up at $t = 1$.