Final Examination: Due on or before May 15, 2002

Q1.

Consider the partial differential equation

$$yu_x(x,y) - xu_y(x,y) = 0$$

in the plane with the nonnegative x-axis removed. The boundary data is specified as $u(x,0) = f(x)$ for $x \geq 0$. What is the solution $u(x,y)$?

Q2.

Suppose we wish to solve the heat equation

$$u_t + \frac{1}{2}u_{xx} \quad \text{on} \quad [0,T] \times [0,\infty).$$

In addition to the value $u(T,x) = f(x)$ we have to specify the Dirichlet boundary data $g(t)$ at $x = 0,\ 0 \leq t \leq T$. We may try to avoid the consideration of the boundary $x = 0$ by a change of variables, replacing x with $y = \log x$ so that we will have $-\infty < y < \infty$. Show that the new equation in (t,y) coordinates takes the form

$$u_t + b(y)u_y + \frac{1}{2}a(y)u_{yy} = 0 \quad \text{on} \quad [0,T] \times (-\infty,\infty)$$

Find a and b explicitly. Why can we not solve uniquely this equation on $[0,\infty) \times R$ which now presumably has no boundary? Where does the theory of SDE break down for

$$dy(t) = b(y(t))dt + \sqrt{a(y(t))}d\beta(t) \quad ?$$

Q3.

Suppose there is a stock that evolves like the geometric Brownian motion

$$dx(t) = \mu x(t)dt + \sigma(x(t)d\beta(t)$$

and a bond that grows at the rate

$$dy(t) = r\,y(t)\,dt$$

with an interest rate of r. The current asset $A(t)$ can be divided between the two investments in the ratio $\pi(t)$ in stock and $(1 - \pi(t))$ in bond, the only retsriction being $0 \leq \pi(t) \leq 1$. There are no transaction costs involved in moving the assets freely between the stock and the bond. If the goal is to maximize the utility

$$E\Big[[A(t)]^\alpha\Big]$$

where $0 < \alpha < 1$ is a measure of risk averseness, what is the optimal strategy? What happens to the strategy as $\alpha \to 0$? How do you interpret the $\alpha = 0$ limit?

Q4.

Consider the recurrence relation for functions $u_n(x)$

$$u_{n+1}(x) = \frac{(1-\delta f(x))}{2\delta}\int_{-\infty}^{\infty} u_n(y)e^{-\frac{|x-y|}{\delta}}dy + \delta g(x)\ ; u_0(x) = 0$$

where $\delta > 0$ is small and $f(x) \geq 0$ and $g(x)$ are nice bounded functions. How will you determine the limit

$$u(t,x) = \lim_{\substack{n\to\infty \\ \delta\to 0 \\ n\delta\to t}} u_n(x)$$

as the solution of a PDE?

Q5.

If

$$dx(t) = b(x(t))dt + \sqrt{a(x(t))}d\beta(t)$$

is an SDE in one dimension, show that the solution $u(a,b,x)$ of

$$\frac{a(x)}{2}u_{xx} + b(x)u_x = 0\ u(a) = 1, u(b) = 0$$

gives for $a < x < b$, the probability that the solution to the SDE starting from x exits from a before exiting from b. Calculate $u(x)$ explicitly for the geometric Brownian Motion

$$dx(t) = \mu x(t)dt + \sigma x(t)d\beta(t)$$

Calculate

$$u(a,x) = \lim_{b\to\infty} u(a,b,x).$$

Under what conditions on μ and σ is it true that

$$\lim_{a\to 0} u(a,x) = 0$$

for any $x > 0$?