2 Singular Integrals

We start with a very useful covering lemma.

Lemma 1. *Suppose $K \subset S$ is a compact subset and I_α is a covering of K. There is a finite subcollection $\{I_j\}$ such that*

1. *$\{I_j\}$ are disjoint.*

2. *The intervals $\{3I_j\}$ that have the same midpoints as $\{I_j\}$ but three times the lenghth cover K.*

Proof. We first choose a finite subcover. From the finite subcover we pick the largest interval. In case of a tie pick any of the competing ones. Then, at any stage, of the remaining intervals from our finite subcollection we pick the largest one that is disjoint from the ones already picked. We stop when we cannot pick any more. The collection that we end up with is clearly disjoint and finite. Let $x \in K$. This is covered by one of the intervals I from our finite subcollection covering K. If I was picked there is nothing to prove. If I is not picked it must intersect some I_j already picked. Let us look at the first such interval and call it I_j. I is disjoint from all the previously picked ones and I was passed over when we picked I_j. Therefore inaddition to intersecting I_j, I is not larger than I_j. Therefore $3I_j \supset I \ni x$. $\square$

This lemma is used in proving maximal inequalities. For instance, for the Hardy-Littlewood maximal function we have

Theorem 1. *Let $f \in L_1(S)$. Define*

$$M_f(x) = \sup_{0<r<\frac{\pi}{2}} \frac{1}{2r} \int_{|y-x|<r} |f(y)|dy \tag{1}$$

$$\mu[x : M_f(x) > \ell] \leq \frac{3\int |f(y)|dy}{\ell} \tag{2}$$

Proof. Let us denote by E_ℓ the set

$$E_\ell = \{x : M_f(x) > \ell\}$$

and let $K \subset E_\ell$ be an arbitrary compact set. For each $x \in K$ there is an interval I_x such that

$$\int_{I_x} |f(y)|dy \geq \ell\mu(I_x)$$

Clearly $\{I_x\}$ is a covering of K and by lemma we get a finite disjoint sub collection $\{I_j\}$ such that $\{3I_j\}$ covers K. Adding them up

$$\int |f(y)|dy \geq \sum_j \mu(I_j) \geq \frac{1}{3}\sum_j \mu(3I_j) \geq \mu(K)$$

Sine $K \subset E_\ell$ is arbitrary we are done. □

There is no problem in replacing $\{x : |M_f(x)| > \ell\}$ by $\{x : |M_f(x)| \geq \ell\}$. Replace ℓ by $\ell - \epsilon$ and let $\epsilon \to 0$.

This theorem can be used to prove the Labesgue diffrentiability theorem.

Theorem 2. *For any $f \in L_1(S)$,*

$$\lim_{h\to 0} \frac{1}{2h}\int_{|x-y|\leq h} |f(y) - f(x)|dy = 0 \qquad \text{a.e.} \quad x \tag{3}$$

Proof. It is sufficient to prove that for any $\delta > 0$

$$\mu[x : \limsup_{h\to 0} \frac{1}{2h}\int_{|x-y|\leq h} |f(y) - f(x)|dy \geq \delta] = 0$$

Given $\epsilon > 0$ we can write $f = f_1 + g$ with f_1 continuous and $\|g\|_1 \leq \epsilon$ and

$$\begin{aligned}\mu[x : \limsup_{h\to 0} \frac{1}{2h}\int_{|x-y|\leq h} |f(y) - f(x)|dy \geq \delta] \\ &= \mu[x : \limsup_{h\to 0} \frac{1}{2h}\int_{|x-y|\leq h} |g(y) - g(x)|dy \geq \delta] \\ &\leq \mu[x : \sup_{h>0} \frac{1}{2h}\int_{|x-y|\leq h} |g(y) - g(x)|dy \geq \delta] \\ &\leq \frac{3\|h\|_1}{\delta} \leq \frac{3\epsilon}{\delta}\end{aligned}$$

Since $\epsilon > 0$ is arbitrary we are done. □

In other words the maximal inequality is useful to prove almost sure convergence. Typically almost sure convergence will be obvious for a dense set and the maximal inequality will be used to interchange limits in the approximation.

Another summability method, like the Fejer sum that is often considred is the Poisson sum

$$S(\rho, x) = \sum_n a_n \rho^{|n|} e^{inx}$$

and the kernel corresponding to it is the Poisson kernel

$$p(\rho, z) = \frac{1}{2\pi} \sum_n \rho^{|n|} e^{inz} = \frac{1}{2\pi} \frac{1-\rho^2}{(1 - 2\rho\cos z + \rho^2)} \tag{4}$$

so that

$$P(\rho, x) = \int f(y) p(\rho, x - y) dy$$

It is left as an exercise to prove that for for $1 \le p < \infty$, every $f \in L_p$ $P(\rho, \cdot) \to f(\cdot)$ in L_p as $\rho \to 1$. We will prove a maximal inequality for the Poisson sum, so that as a consequence we will get the almost sure convergence of $P(\rho, x)$ to f for every f in L_1.

Theorem 3. *For every f in L_1*

$$\mu[x : \sup_{0 \le \rho < 1} P(\rho, x) \ge \ell] \le \frac{C\|f\|_1}{\ell} \tag{5}$$

Proof. The proof consists of estimating the Poisson maximal function in-terms of the Hardy-Littlewood maximal function $M_f(x)$. We begin with some simple estimates for the Poisson kernel $p(\rho, z)$.

$$\begin{aligned} p(\rho, z) =& \frac{1}{2\pi} \frac{1-\rho^2}{(1-\rho)^2 + 2\rho(1 - \cos z)} \le \frac{1}{2\pi} \frac{1-\rho^2}{(1-\rho)^2} \\ =& \frac{1}{2\pi} \frac{1+\rho}{1-\rho} \le \frac{1}{\pi} \frac{1}{1-\rho} \end{aligned}$$

The problem therefore is only as $\rho \to 1$. Lets us assume that $\rho \ge \frac{1}{2}$.

For any symmetric function $\phi(z)$ the intgral

$$
\begin{aligned}
&|\int_{-\pi}^{\pi} f(z)\phi(z)dz| \\
&\qquad = |\int_0^{\pi} [f(z)+f(-z)]\phi(z)dz| \\
&\qquad = |\int_0^{\pi} \phi(z)[\frac{d}{dz}\int_{-z}^{z} f(y)dy]dz| \\
&\qquad \leq |\int_0^{\pi} \phi'(z)[\int_{-z}^{z} f(y)dy]dz| + |\phi(\pi)|\int_{-\pi}^{\pi} f(z)dz| \\
&\qquad \leq \int_0^{\pi} 2|z\phi'(z)|[\int |f(y)|\lambda_z(dy)]dz + \phi(\pi)|\int_{-\pi}^{\pi} |f(z)|dz \\
&\qquad \leq 2M_f(0)\int_0^{\pi} |z\phi'(z)|dz + \phi(\pi)||M_f(0)|
\end{aligned}
$$

For the Poisson kernel

$$
\begin{aligned}
|z\frac{d}{dz}p(\rho,z)| &= \frac{1}{2\pi}\frac{1-\rho^2}{(1-2\rho\cos z+\rho^2)^2}2\rho|z\sin z| \\
&\leq \frac{1}{\pi}\frac{(1-\rho)z^2}{(1-\rho)^4+(1-\cos z)^2} \\
&\leq C\frac{(1-\rho)z^2}{(1-\rho)^4+z^4}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{-\pi}^{\pi} |z\frac{d}{dz}p(\rho,z)|dz &\leq C\int_{-\pi}^{\pi}\frac{(1-\rho)z^2}{(1-\rho)^4+z^4}dz \\
&= \int_{-\frac{\pi}{1-\rho}}^{\frac{\pi}{1-\rho}} \frac{z^2}{1+z^4}dz \\
&\leq \int_{-\infty}^{\infty}\frac{z^2}{1+z^4}dz \leq C_1
\end{aligned}
$$

uniformly in ρ.

$\square$

Interpolation theorems play a very important role in Harmonic Analysis. An example is the following

Theorem 4 (Marcinkiewicz). *Let T be a sublinear map defiened on $L_p \cap L_q$ that satisfies weak type inequlities*

$$\mu[|x| : |(Tf)(x)| \geq \ell] \leq \frac{C_i \|f\|_{p_i}^{p_i}}{\ell^{p_i}} \tag{6}$$

for $i = 1, 2$ where $1 \leq p_1 < p_2 < \infty$. Then for $p_1 < p < p_2$, there are constants C_p such that

$$\|Tf\|_p \leq C_p \|f\|_p \tag{7}$$

Note that T need not be linear. It need only satisfy

$$|T(f+g)|(x) \leq |Tf|(x) + |Tg|(x) \tag{8}$$

Proof. Let $p \in (p_1, p_2)$ be fixed. For any function $f \in L_p$ and for any positive number a we deine $f_a = f\chi_{\{|f| \leq a\}}$ and $f^a = \chi_{\{|f|>a\}}$. Clearly $f_a \in L_{p_2}$ and $f^a \in L_{p_1}$

$$\begin{aligned}\mu[x : |Tf(x)|| \geq 2\ell] &\leq \mu[x : |Tf_a(x)|| \geq \ell] + \mu[x : |Tf^a(x)|| \geq \ell] \\ &\leq \frac{C_2}{\ell^{p_2}} \int_{|f(x)| \leq a} |f(x)|^{p_2} d\mu + \frac{C_1}{\ell^{p_1}} \int_{|f(x)|>a} |f(x)|^{p_1} d\mu\end{aligned}$$

Take $a = \ell$, multiply by ℓ^{p-1} and integrate with respect to ℓ from 0 to ∞. Use Fubini's theorem. We get

$$\int_0^\infty \ell^{p-1} \mu[x : |Tf(x)|| \geq 2\ell] d\ell \leq [\frac{C_2}{p_2 - p} + \frac{C_1}{p - p_1}] \int |f(x)|^p d\mu \tag{9}$$

Since the left hand side is $\frac{\|Tf\|_p^p}{p}$ we are done. ∎

There is a slight variation of the argument that allows p_2 to be infinite provided T is bounded on L_∞. If we denote the norm by C_2 we use

$$\mu[x : |Tf(x)|| \geq (C+1)\ell] \leq \mu[x : |Tf^a(x)|| \geq \ell]$$

and proceed as before.

A different interpolation theorem for **linear** maps T is the following

Theorem 5 (Riesz-Thorin). *If a linear map T is bounded from L_{p_i} into L_{p_i} with a bound C_i for $i = 1, 2$ then for $p_1 \leq p \leq p_2$ it is bounded from L_p into L_p with a bound C_p that can be taken to be*

$$C_p = C_1^t C_2^{1-t} \tag{10}$$

where t is determined by

$$\frac{1}{p} = t\frac{1}{p_1} + (1-t)\frac{1}{p_2} \tag{11}$$

Proof. The proof uses methods from the theory of functions of a complex variable. The starting point is the maximum modulus principle. Let us assume that $u(z)$ is analytic in the open strip $a < Rez < b$ and bounded and continuous in the closed strip $a \leq Rez \leq b$. Let $M(x)$ be the maximum modulus of the function on the line $Rez = x$. Then $\log M(x)$ is a convex function of x.This is not hard to see. Clearly the maximum principle dictates that

$$M(x) \leq \max[M(a), M(b)]$$

If one is worried about the maximum being attained, one can always mutiply by $e^{\epsilon z^2}$ and let ϵ go to 0. Replacing $u(z)$ by $u(z)e^{tz}$ yields the inequality

$$M(x) \leq \max[M(a)e^{t(a-x)}, M(b)e^{t(b-x)}]$$

optimizing with respect to t we get,

$$M(x) \leq \max[[M(a)]^{\frac{b-x}{b-a}}, [M(b)]^{\frac{x-a}{b-a}}]$$

which is the required convexity.

We note that the maximum of any collection of convex functions is again convex. The proof is completed by representing $\log F(p)$, where $F(p)$ is the norm of T from L_p to L_p, as the supremum of a bunch of functions that are convex in $x = \frac{1}{p}$.

$$\begin{aligned}
\|T\|_{p,p} &= \sup_{\substack{\|f\|_p \le 1 \\ \|g\|_q \le 1}} \left| \int g(Tf) d\mu \right| \\
&= \sup_{\substack{\|f\|_p \le 1, f \ge 0, |\phi|=1 \\ \|g\|_q \le 1, g \ge 0, |\psi|=1}} \left| \int (g\psi)(T(f\phi)) d\mu \right| \\
&= \sup_{\substack{\|f\|_1 \le 1, f > 0, |\phi|=1 \\ \|g\|_1 \le 1, g > 0, |\psi|=1}} \left| \int (g^x\psi)(T(f^{1-x}\phi)) d\mu \right| \\
&= \sup_{\substack{\|f\|_1 \le 1, f > 0, |\phi|=1 \\ \|g\|_1 \le 1, g > 0, |\psi|=1 \\ Rez=x}} \left| \int (g^z\psi)(T(f^{1-z}\phi)) d\mu \right| \\
&= \sup_{\substack{\|f\|_1 \le 1, f > 0, |\phi|=1 \\ \|g\|_1 \le 1, g > 0, |\psi|=1}} \sup_{Rez=x} |u(f, g, \phi, \psi, z)|
\end{aligned}$$

$\square$

In particular for the Hardy-Littlewood or Poisson maximal function the L_∞ bound is trivial and we now have a bound for the L_p norm of the maximal function in terms of the L_p norm of the original function provided $p > 1$.

For a convolution operator of the form

$$(Tf)(x) = \int_{-\pi}^{\pi} f(y)k(x-y)dy \tag{12}$$

we saw that for it to be bounded as an operator from L_1 into itself we need k to be in L_1. However for $1 < p < \infty$ the operator can some times be bounded even if k is not in L_1. This is proved by establishing a bound from L_2 to L_2 and a weak type inequality in L_1. We can then use Marcinkiewicz interpolation, followed by Riesz-Thorin interpolation.

Theorem 6. *If*

$$\hat{k}(n) = \int e^{inz} k(z) dz$$

is bounded in absolute value by C, then the convolution operator given by equation (12) is bounded by C as an operator from L_2 to L_2.

Proof. Use the the orthonormal basis e^{inx} to diagonalize T

$$Te^{inx} = \hat{k}(n)e^{inx} \tag{13}$$

□

We now proceed to establish weak type $(1, 1)$ estimate. We shall assune that we have a kernel k in L_1 that satisfies

1. $$\sup_n |\int k(y)e^{iny}dy| = C_1 < \infty \tag{14}$$

2. $$\sup_y \int_{x:|x-y|>2|y|} |k(x-y) - k(x)|dx = C_2 < \infty \tag{15}$$

Although we have assumed that k is in L_1 we will prove a weak type $(1, 1)$ bound.

Theorem 7. *The operator of convolution by* k

$$(T_k f)(x) = \int_{-\pi}^{\pi} k(x-y)f(y)dy \tag{16}$$

satisfies the weak type inequality (1,1)

$$\mu[|x| : |(Tf)(x)| \geq \ell] \leq \frac{C}{\ell}\|f\|_1 \tag{17}$$

with a constant C *that depends only on* C_1 *and* C_2*.*

Proof. Proof involves several steps.

- First we observe that the Hardy-Littlewood maximal function given by (1) satisfies equation 2). The set $G = [x : M_f(x) \geq \ell]$ is an open set in $[-\pi, \pi]$ and has Lebsgue measure atmost $\frac{3\|f\|_1}{\ell}$. We assume that $\ell > \frac{3\|f\|_1}{2\pi}$ so that $B = G^c$ is nonempty. We write the open set G as a possible countable union of **disjoint** open intervals I_j of length r_j centered at x_j. Note that the end points $x_j \pm \frac{1}{2}r_j$ necessarily belong to B. The maximal inequality assures us that

$$\sum_j r_j \leq \frac{3\|f\|_1}{\ell}$$

- Let us define the averages

$$m_j = \frac{1}{r_j} \int_{I_j} f(y) dy$$

and write f in the form

$$\begin{aligned} f(x) &= [f(x)1_B(x) + \sum_j m_j 1_{I_j}(x)] + \sum_j [f(x) - m_j] 1_{I_j}(x) \\ &= g(x) + \sum_j h_j(x) \end{aligned}$$

- We have the bounds

$$\begin{aligned} |m_j| &\leq \frac{1}{r_j} \int_{I_j} |f(y)| dy \leq \frac{1}{r_j} \int_{\tilde{I}_j} |f(y)| dy \\ &\leq 2 \frac{1}{2r_j} \int_{\tilde{I}_j} |f(y)| dy \leq 2 M_f(x_j \pm r_j) \leq 2\ell \end{aligned}$$

Here $\tilde{I}_j$ is the interval centered around $x_j \pm \frac{r_j}{2}$ of length $2r_j$. In particular $\|g\|_\infty \leq 2\ell$. On the other hand

$$\sum_j \|h_j\|_1 = \sum_j \int_{I_j} |f(y) - m_j| dy \leq 2 \sum_j \int_{I_j} |f(y)| dy \leq 2\|f\|_1$$

We therefore have

$$\|g\|_1 \leq 3\|f\|_1$$

Note that the decomposition depends on ℓ. Let us write the corresponding sum

$$u = T_k f = T_k g + \sum_j T_k h_j = v + \sum_j w_j = v + w$$

- We estimate the L_2 norm of v and the L_1 norm of w on large enough set. Then use Tchebychev's inequality.

$$\mu[x : |v(x)| \geq \frac{\ell}{2}] \leq \frac{\|v\|_2^2}{\ell^2} \leq \frac{C_1 \|g\|_2^2}{\ell^2} \leq \frac{2\ell C_1 \|g\|_1}{\ell^2} = \frac{6 C_1 \|f\|_1}{\ell}$$

Let us denote by $\hat{I}_j$ the interval centered around x_j of length $3r_j$ and by $U = \cup_j \hat{I}_j$. We begin by estimating $\|w.1_{U^c}\|_1$.

$$\begin{aligned}
\|w.1_{U^c}\|_1 &\leq \int_{U^c} \sum_j |\int_{I_j} k(x-y)[f(y)-m_j]dy|dx \\
&= \int_{U^c} \sum_j |\int_{I_j} [k(x-y)-k(x-x_j)][f(y)-m_j]dy|dx \\
&\leq \int_{U^c} \sum_j \int_{I_j} |k(x-y)-k(x-x_j)||f(y)-m_j|dydx \\
&= \sum_j \int_{I_j} |f(y)-m_j|dy \int_{U^c} |k(x-y)-k(x-x_j)|dx \\
&\leq \sum_j \int_{I_j} |f(y)-m_j|dy \int_{\hat{I}_j^c} |k(x-y)-k(x-x_j)|dx \\
&\leq \sum_j \int_{I_j} |f(y)-m_j|dy \int_{x:|x-y|\geq 2|y-x_j|} |k(x-y)-k(x-x_j)|dx \\
&\leq C_2 \sum_j \int_{I_j} |f(y)-m_j|dy \\
&\leq 2C_2\|f\|_1
\end{aligned}$$

We have used here two facts. $f(y)-m_j$ has mean zero on I_j . If $y \in I_j$ and $x \in \tilde{I}_j^c$, then $|y-x| \geq r_j \geq 2|y-x_j|$. On the other hand

$$\mu(U) \leq \sum \mu(\tilde{I}_j) \leq 3\sum \mu(I_j) = 3\sum_j r_j \leq \frac{9\|f\|_1}{\ell}$$

- Finally we can put the pieces together.

$$\begin{aligned}
\mu(x:|u(x)| \geq 2\ell) &\leq \mu(x:|v(x)| \geq \ell) + \mu(x:|w(x)| \geq \ell) \\
&\leq \frac{6C_1\|f\|_1}{\ell} + \frac{9\|f\|_1}{\ell} + \frac{2C_2\|f\|_1}{\ell}
\end{aligned}$$

or

$$\mu(x:|u(x)| \geq \ell) \leq \frac{(12C_1+18+4C_2)\|f\|_1}{\ell} = \frac{C\|f\|_1}{\ell}$$

$\square$

There is one point that we should note. For the interval doubling construction on the circle we should be sure that we do not see for instance any interval of lenghth larger than $\frac{\pi}{2}$ in G. This can be ensured if we take $\ell > \frac{6\|f\|_1}{\pi}$. The inequality is however satisfied for all ℓ because $C \geq 12$.

We want to look at the special kernel $k(y) = \frac{1}{y}$ which is not in L_1. We consider its truncation

$$k_\delta(y) = \frac{1}{y}\mathbf{1}_{\{|y|\geq\delta\}}(y)$$

First we estimate the Fourier transform

$$\begin{aligned} |\int_{|y|\geq\delta} \frac{e^{iny}}{y} dy| &= 2|\int_\delta^\pi \frac{\sin ny}{y} dy| \\ &= 2|\int_{n\delta}^{n\pi} \frac{\sin y}{y} dy| \leq 4 \sup_{0<a<\infty} |\int_0^a \frac{\sin y}{y} dy| \leq C_1 \end{aligned}$$

Next in order to verify the condition (15) we need to estimate the following quantity uniformly in y and δ.

$$\int_{x:|x-y|>2|y|} |k_\delta(x-y) - k_\delta(x)| dx$$

There are three sets over which the integral does not vanish.

$$\begin{aligned} F_1 &= \{x : |x-y| > 2|y|, |x-y| \geq \delta, |x| \geq \delta\} \\ F_2 &= \{x : |x-y| > 2|y|, |x-y| \leq \delta, |x| \geq \delta\} \\ F_3 &= \{x : |x-y| > 2|y|, |x-y| \geq \delta, |x| \leq \delta\} \end{aligned}$$

We consider

$$\begin{aligned} \int_{F_1} |\frac{1}{x-y} - \frac{1}{x}| dx &\leq \int_{x:|x-y|\geq 2|y|} |\frac{1}{x-y} - \frac{1}{x}| dx \\ &\leq \int_{|z-1|\geq 2} |\frac{1}{z-1} - \frac{1}{z}| dz \\ &= C_3 \end{aligned}$$

It is clear that $F_2 \subset [-2\delta, 2\delta]$. Therefore

$$\int_{F_2} \frac{1}{|x|} dx \le 2 \int_{\delta}^{2\delta} \frac{dx}{x} = C_4$$

Finally $F_3 \subset [x : |x-y| \le 2\delta]$ and works similarly. With $C_2 = C_3 + 2C_4$ we are done.

We are now ready to prove

Theorem 8. *For any $f \in L_p$ the partial sums $s_N(f,x)$ converge to f in L_p provided $1 < p < \infty$.*

Proof. We need only prove, for $1 < p < \infty$, a bound from L_p to L_p, for the partial sum operators

$$(T_N f)(x) = \int f(x-y) k_N(y) dy$$

with

$$k_N(z) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})z}{\sin\frac{z}{2}}$$

that is uniform in N. In terms of multipliers we are looking at a uniform L_p bound for the operators defined by

$$\hat{k}_N(n) = \mathbf{1}_{\{|n| \le N\}}(n)$$

Let us define the operators M_k as multiplication by e^{ikx} which are isometries in every L_p. P_0 is the operator of projection to constants, i.e. the operator with multiplier $\mathbf{1}_{\{0\}}(n)$ which is clearly bounded in every L_p. Finally the Hilbert transform S is the one with multiplier *signum* n. It is easy to verify that

$$T_N = M_{-N}\frac{1}{2}[(S+I)+P_0]M_N - M_{(N+1)}]\frac{1}{2}[(S+I)+P_0]M_{-(N+1)}]$$

This reduces the problem to proving that a single operator S is bounded on L_p. The kernel is calculated to be

$$s(z) = \frac{1}{2\pi} \cot\frac{z}{2}$$

This can be replaced by the modified kernel

$$k(z) = \frac{1}{\pi z}$$

and we are done. ☐