Limit Theorems.

Set 4. Due Oct 24, 2002.

Q1. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed random variables taking the values ± 1, each with probability $\frac{1}{2}$. For $a > 0$, show that the probability

$$p_n(a) = P[\frac{X_1 + \cdots + X_n}{n} \geq a]$$

goes to 0 geometrically and calculate

$$\rho(a) = \lim_{n \to \infty} [p_n(a)]^{\frac{1}{n}}$$

as a function of a.

Q2. More generally if the common distribution α of $X_1, X_2 \ldots$ has the properties

$$M(\lambda) = \int e^{\lambda x} d\alpha(x) < \infty$$

for all $\lambda \geq 0$, and

$$P[X \geq x] > 0$$

for every $x \in R$, then show that for any $a > E[X]$

$$\rho(a) = \lim_{n \to \infty} [p_n(a)]^{\frac{1}{n}} = \lim_{n \to \infty} \left[P[\frac{X_1 + \cdots + X_n}{n} \geq a] \right]^{\frac{1}{n}} = \inf_{\lambda > 0} e^{-a\lambda} M(\lambda)$$