Chapter 16

Reflected Brownian Motion

Often we encounter Diffusions in regions with boundary. If the process can reach the boundary from the interior in finite time with positive probability we need to decide what we want to do afterwards. If allowed to continue the process may not remain inside the domain or even inside the closed domain including the boundary. There are some choices. We could freeze it at the point where it exited. Or define some motion on the boundary that it can follow. Is there a way where we can force it to return to the interior and still remain a Markov process with continuous trajectories.

Reflected Brownian motion on the half line $[0, \infty)$ is a way of keeping Brownian motion in the half line $[0, \infty)$. It can be defined as the unique process P_x supported on $\Omega = C[[0, \infty); [0, \infty)]$ with the following properties:

1. It starts from $x(0) = x$, i.e.

$$P_x[x(0) = x] = 1 \tag{16.1}$$

2. It behaves locally Brownian motion on $(0, \infty)$. This is formulated in the following manner. If τ is any stopping time and $\sigma = \inf\{t : t \geq \tau, x(t) = 0\}$ is the hitting time of 0 after τ then the conditional probability distribution of P_x given $\mathcal{F}_\tau$ agrees with $Q_{x(\tau)}$ the distribution of the Brownian motion starting at $x(\tau)$ on the $\sigma-$field $\mathcal{F}_\sigma^{\tau(\omega)}$.

3. Finally the time spent at the boundary is 0. i.e. for any t,

$$E^{P_x}[\int_0^t \mathbf{1}_{\{0\}}(x(s))ds] = 0 \tag{16.2}$$

Lemma 16.1. *Behaving locally like the Brownian motion is equivalent to the following condition:*

$$f(x(t)) - f(x(0)) - \frac{1}{2}\int_0^t f_{xx}(x(s))ds \tag{16.3}$$

is a martingale for any bounded smooth function f that is a constant (which can be taken to be 0 *with out loss of generality) in some neighborhood of* 0*.*

Proof. The equivalence is easy to establish. Let us take $\epsilon > 0$ and consider the exit time τ_ϵ from (ϵ, ∞). Any smooth function on $[\epsilon, \infty)$ can be extended to $[0, \infty)$ in such a way that it has compact support in $(0, \infty)$. Therefore the martingale property (16.3) holds for arbitrary smooth bounded f until τ_ϵ. Hence P_x agrees with Q_x on $\mathcal{F}_{\tau_\epsilon}$. But $\epsilon > 0$ is arbitrary and we can let it go to 0. $\mathcal{F}_{\tau_\epsilon} \uparrow \mathcal{F}_\tau$ as $\epsilon \to 0$. In the other direction if f is supported on $[a, \infty)$ we can consider the sequence of stopping times defined successively, starting from $\tau_0 = 0$ and for $j \geq 0$,

$$\tau_{2j+1} = \inf\{t : t \geq \tau_{2j}, x(t) = 0\}$$

and for $j \geq 1$,

$$\tau_{2j} = \inf\{t : t \geq \tau_{2j-1}, x(t) = a\}$$

Clearly

$$Z(t) = f(x(t)) - f(x(0)) - \frac{1}{2}\int_0^t f_{xx}(x(s))ds$$

is a martingale in $[\tau_{2j}, \tau_{2j+1}]$ and is constant in $[\tau_{2j-1}, \tau_{2j}]$ when $x(t) \in [0, a]$. It is now easy to verify that $Z(t)$ is a martingale. Just write for $s < t$,

$$Z(t) - Z(s) = \sum_{j \geq 0}[Z((\tau_{j+1} \vee s) \wedge t) - Z((\tau_j \vee s) \wedge t)]$$

□

Theorem 16.2. *There is a unique P_x satisfying (16.1), (16.2) and (16.3). It can also be characterized by replacing (16.2) and (16.3) by the following. f is a smooth bounded function satisfying $f'(0) \geq 0$, then*

$$Z_f(t) = f(x(t)) - f(x(0)) - \frac{1}{2}\int_0^t f_{xx}(x(s))ds \tag{16.4}$$

is a sub-martingale with respect to $(\Omega, \mathcal{F}_t^0, P_x)$. In particular if $f'(0) = 0$, $Z_f(t)$ is a martingale. If $\beta(t)$ is the ordinary Brownian motion starting from x, then P_x is the distribution of $x(t) = |\beta(t)|$ and is a Markov process with transition probability density,

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}}\left[e^{\frac{(y-x)^2}{2t}} + e^{\frac{y+x)^2}{2t}}\right]$$

for $t > 0, x, y \geq 0$.

Remark 16.1. In both (16.3) and (16.4) the condition that f is bounded can be dropped and replaced by f having polynomial growth. If f is x^{2n} for large x, then f_{xx} is $c_n x^{2n-2}$. If $n = 1$

$$E[[x(t)]^2] = E[[x(0)]^2] + \int_0^t 1\ ds$$

This is easily justified by standard methods of approximating x^2 by bounded functions below it. By induction on n, one can show in a similar fashion that

$$E[[x(t)]^{2n}] = E[[x(0)]^{2n}] + n(2n-1)\int_0^t [x(s)]^{2n-2}\, ds$$

and deduce bounds of the form $E\big[[x(t)]^{2n}\big] \le C_n[x^{2n} + t^n]$ for all n. Then the martingale property will extend to smooth functions with polynomial growth.

Proof. First let us establish the equivalence between (16.2)-(16.3) and (16.4).

Given a function $f(x)$ with $f'(0) = 0$, we can approximate it with functions that are constant near 0. $f^{(n)}(x) = f(0)$ for $0 \le x \le \frac{1}{n}$ and $f^{(n)}(x) = f(x - \frac{1}{n})$. It is piecewise smooth, with matching first derivatives. Therefore it is easy to check that

$$f^{(n)}(x(t)) - f^{(n)}(x(0)) - \int_0^t f^{(n)}_{xx}(x(s))ds$$

is a Martingale. Let $n \to \infty$, then $f^{(n)} \to f$ and $f^{(n)}_{xx} \to f_{xx}$ except at 0. But $f^{(n)}_{xx}$ is uniformly bounded and because of (16.2) we obtain (16.4) for functions with $f'(0) = 0$. Conversely if $Z(t)$ is a martingale when $f'(0) = 0$, we can then show that (16.2) holds. Consider the function

$$f_\epsilon(x) = \begin{cases} x^2 & \text{if } x \le \epsilon \\ 2\epsilon x - \epsilon^2 & \text{if } x \ge \epsilon \end{cases} \tag{16.5}$$

It is piecewise smooth, with matching first derivatives. Therefore

$$E[f_\epsilon(x(t))] - f_\epsilon(x) = \frac{1}{2}E[\int_0^t f''_\epsilon(x(s))ds] = E[\int_0^t \mathbf{1}_{[0,\epsilon]}(x(s))ds]$$

Since $f_\epsilon(x) \le 2\epsilon x$ it follows that

$$\lim_{\epsilon \to 0} E[\int_0^t \mathbf{1}_{[0,\epsilon]}(x(s))ds] = 0$$

proving (16.2).

To prove the sub-martingale property, let $\phi(x)$ be a smooth, bounded function that is identically equal to x in some interval $[0, \delta]$. Then $g(x) = f(x) - f'(0)\phi(x)$ satisfies $g'(0) = 0$ and therefore Z_g is a martingale. It is then enough to prove that $Z_\phi(t)$ is sub-martingale. We repeat an argument used earlier in the proof of Lemma 16.1 of writing

$$Z_\phi(t) = \sum_j [Z_\phi(\tau_{j+1} \wedge t) - Z_\phi(\tau_j \wedge t)]$$

with $a = \delta$. The terms with even j are martingales and as for terms with odd j, $\phi(x(\tau_{j+1} \wedge t)) - \phi(x(\tau_j \wedge t)) \ge 0$. An easy calculation now shows that

$$E^{P_x}[Z_\phi(t) - Z_\phi(s)|\mathcal{F}_s] \ge 0$$

Finally we will identify any P_x satisfying (16.1), (16.2) and (16.3) or (16.4), has the distribution of $x(t) = |\beta(t)|$ starting from $\beta(0 + x$. $p(t, x, y)$ is clearly the transition probability of $|\beta(t)|$ which is Markov because Brownian motion has the $x \to -x$ symmetry. The function

$$u(t, x) = \int q(t, x, y) f(y) dy$$

satisfies

$$u_t = \frac{1}{2} u_{xx}; u(0, x) = f(x), u_x(t, 0) = 0$$

Just as in the case of diffusions with out boundary, if

$$f(x(t)) - f(x(0)) - \frac{1}{2} \int_0^t f_{xx}(x(s)) ds$$

is a martingale for all smooth f satisfying $f'(0) = 0$, then it follows that for all smooth functions $v = v(s, x)$ satisfying $v_x(s, 0) = 0$ for all s,

$$v(t, x(t)) - v(0, x(0)) - \int_0^t [v_s(s, x(s)) + \frac{1}{2} v_{xx}(0, x(s))] ds$$

is a martingale. Therefore $u(T-t, x(t))$ is a martingale under P_x and $E^{P_x}[f(x(T))] = u(T, x)$. This identifies the marginal distributions of P_x as that of $|\beta(\cdot)$. Regular conditioning argument allows us to complete the identification. □

The reflected Brownian motion can be defined in terms of the ordinary Brownian motion $\beta(t)$ as a solution of the equation $x(t) \geq 0$

$$x(t) = x + \beta(t) + A(t)$$

where $A(t)$ is continuous and increasing, but is allowed to increase only when $x(s) = 0$, i.e

$$\int_{\{s: x(s) > 0\}} dA(s) = 0$$

Such a solution is unique. If $x_i(t); i = 1, 2$ are two solutions

$$\begin{aligned} d[x_1(t) - x_2(t)]^2 &= 2(x_1(t) - x_2(t)) d(A_1(t) - A_2(t)) \\ &= -2x_1(t) dA_2(t) - 2x_2(t) dA_1(t) \leq 0 \end{aligned}$$

But $x_1(0) = x_2(0)$. Implies $x_1(t) \equiv x_2(t)$ for all t.

Existence is easy.

$$x(t) = x + \beta(t) - \inf_{0 \leq s \leq t} [(x + \beta(s))^-]$$

is clearly non negative. $A(t) = -\inf_{0 \leq s \leq t} [(x + \beta(s))^-]$ is increasing and does so only when $[x + \beta(t)]^- = -A(t)$, i.e. $x(t) = 0$. The reflected Brownian motion comes with a representation

$$x(t) = \beta(t) + A(t)$$

where $A(t)$ is a local time at 0, almost surely continuous in t, acting as an infinitely strong drift when $x(t) = 0$ which is a set of times of Lebesgue measure 0. This leads to an Ito's formula of the form

$$\begin{aligned} du(t, x(t)) &= u_t(t, x(t))dt + u_x(t, (x(t))dx(t) + \frac{1}{2}u_{xx}(t, (x(t))dt \\ &= u_t(t, x(t))dt + u_x(t, (x(t))d\beta(t) + u_x(t, 0)dA(t) + \frac{1}{2}u_{xx}(t, (x(t))dt \end{aligned}$$

which is a way of seeing the effect of the boundary condition satisfied by u.

There is a way of slowing down the process at the boundary. Let

$$\sigma(s) = \rho A(s) + s$$

Define increasing family of stopping times $\{\tau_t\}$ by

$$\sigma(\tau_t) = t; \quad \tau_{\sigma(s)} = s$$

which has a unique solution which is strictly increasing and continuous in t. Since ds and $dA(s)$ are orthogonal (A(s) increases only when $x(s) = 0$ which is a set of 0 Lebesgue measure), we have

$$dA(s) = \rho^{-1}\mathbf{1}_{\{0\}}(x(s))d\sigma(s)$$

and

$$ds = \mathbf{1}_{\{(0,\infty)\}}(x(s))d\sigma(s)$$

We can now define a new process

$$y(t) = x(\tau_t)$$

We will identify the martingales that go with this process. By Doob's optional stopping theorem,

$$\begin{aligned} Z_f(t) &= f(x(\tau_t)) - f(x(0)) - \frac{1}{2}\int_0^{\tau_t} f''(x(s))ds - f'(0)A(\tau_t) \\ &= f(x(\tau_t)) - f(x(0)) - \frac{1}{2}\int_0^{\tau_t} f''(x(s))\mathbf{1}_{(0,\infty)}(x(s))d\sigma(s) \\ &\qquad - \rho^{-1}f'(0)\int_0^{\tau_t} \mathbf{1}_{\{0\}}(x(s))d\sigma(s) \\ &= f(y(t)) - f(y(0)) - \frac{1}{2}\int_0^t f''(y(s))\mathbf{1}_{(0,\infty)}(y(s))ds \\ &\qquad - \rho^{-1}f'(0)\int_0^t \mathbf{1}_{\{0\}}(y(s))ds \end{aligned}$$

is a martingale. The new process is then a Markov process with generator

$$(\mathcal{L}f)(x) = \begin{cases} \frac{1}{2}f''(x) & \text{if } x > 0 \\ \rho^{-1}f'(0) & \text{if } x = 0 \end{cases}$$

We can calculate the Laplace transform in t of the probability $P_x[x(t) = 0]$. This requires us to solve the resolvent equation, denoting ρ^{-1} by γ

$$\lambda\psi(x) - \frac{1}{2}\psi''(x) = 0, \lambda\psi - \gamma\psi'(0) = 1$$

which can be done explicitly to give

$$\int_0^\infty e^{-\lambda\, t} p(t, x, \{0\}) dt = \frac{1}{\lambda + \gamma\sqrt{2\lambda}} e^{-\sqrt{2\lambda}\, x}$$

In particular $p(t, 0, \{0\}) \to 1$ as $t \to 0$. But $1 - p(t, 0, \{0\})$ does not go to 0 linearly in t. $\{0\}$ is an instantaneous state. As $\gamma \to \infty$ this becomes the reflected process and as $\gamma \to 0$ the absorbed process. For the characterization of these processes $\{P_{\gamma,x}\}$ through martingales we have the following theorem.

Theorem 16.3. *The process $\{P_{\gamma,x}\}$ is characterized uniquely as the process with respect to which*

$$u(t, x(t)) - u(0, x(0)) - \int_0^t [u_s(s, x(s)) + \frac{1}{2} u_{xx}(s, x(s))] \mathbf{1}_{(0,\infty)}(x(s)) ds \quad (16.6)$$

is a sub-martingale provided $(u_t + \gamma u_x)(t, 0) \geq 0$ for all t.

Proof. First let us prove that for any smooth bounded $u(t, x)$,

$$u(t, x(t)) - u(0, x(0)) - \int_0^t u_s(s, x(s)) ds - \int_0^t \frac{1}{2} u_{xx}(s, x(s)) \mathbf{1}_{(0,\infty)}(x(s)) ds$$
$$- \gamma \int_0^t u_x(s, 0) \mathbf{1}_{\{0\}}(x(s)) ds$$

is a martingale. We saw earlier that the time changed process P_x has the property that

$$f(x(t)) - f(x(0)) - \int_0^t [\gamma f'(0) \mathbf{1}_{\{0\}}(x(s)) + \frac{1}{2} f_{xx}(x(s)) \mathbf{1}_{(0,\infty)}(x(s))] ds$$

is a martingale for smooth bounded functions. It follows from this that for smooth functions $u = u(t, x)$,

$$u(t, x(t)) - u(0, (x(0)) - \int_0^t u_s(s, x(s)) ds$$
$$- \int_0^t [\gamma u_x(s, x(s)) \mathbf{1}_{\{0\}}(x(s)) + \frac{1}{2} u_{xx}(s, x(s)) \mathbf{1}_{(0,\infty)}(x(s))] ds$$

is a martingale. If $u_s(s, 0) + \gamma u_x(s, 0) = 0$ for all s, we can replace $\int_0^t u_s(s, x(s)) ds$ by

$$\int_0^t u_s(s, 0) \mathbf{1}_{\{0\}}(x(s)) ds + \int_0^t u_s(s, x(s)) \mathbf{1}_{(0,\infty)}(x(s)) ds$$
$$= -\gamma \int_0^t u_x(s, 0) \mathbf{1}_{\{0\}}(x(s)) ds + \int_0^t u_s(s, x(s)) \mathbf{1}_{(0,\infty)}(x(s)) ds$$

proving that the expression (16.6) is a martingale. On the other hand, if $u_s(s,0)+\gamma u_x(s,0)=h(s)$, then $v(s,x)=u(s,x)-H(s)$ with $H'(s)=h(s)$, satisfies the boundary condition $v_s(s,0)+\gamma v_x(s,0)=0$ and

$$v(t,x(t))-v(0,x(0))-\int_0^t [v_s(s,x(s))ds+\frac{1}{2}v_{xx}(s,x(s))]\mathbf{1}_{(0,\infty)}(x(s))ds \\ -\gamma\int_0^t v_x(s,0)\mathbf{1}_{\{0\}}(x(s))ds$$

is a martingale. Substituting $v(s,x)=u(s,x)-H(s)$ with $H(s)$ nondecreasing, we obtain the result. Finally will will identify the processes $P_{\gamma,x}$ through their associated ODEs.

Consider the solution $g(x)$ of

$$\lambda g(x)-\frac{1}{2}g''(x)=f(x)\quad\text{for } x>0$$
$$\lambda g(0)-\gamma g'(0)=a$$

Then with $u(t,x)=e^{-\lambda\, t}g(x)$ we see that

$$e^{-\lambda t}g(x(t))-g(x(0))+\int_0^t e^{-\lambda\, s}[f(x(s))\mathbf{1}_{(0,\infty)}(x(s))ds+a\mathbf{1}_{(0,\infty)}(x(s))]ds$$

is a martingale. Equating expectations at $t=0$ and $t=\infty$ we identify

$$g(x)=E_x\big[\int_0^\infty e^{-\lambda\, s}[f(x(s))\mathbf{1}_{(0,\infty)}(x(s))ds+a\mathbf{1}_{(0,\infty)}(x(s))]ds\big]$$

□

The situation corresponding to $\frac{1}{2}a(x)D_x^2+b(x)D_x$ is similar so long as $a(x)\geq c>0$. Girsanov formula works and b can be taken care of and one can do random time change as well and reduce the problem to $a=1$, $b=0$.

Markov Chain approximations. We have a transition probability $\pi_h(x,dy)$. We assume that for any $\delta>0$,

$$\sup_x \pi_h(x,\{y:|x-y|\geq\delta\})=o(h)$$

uniformly in x. This implies immediately that the processes $P_{h,x}$ of piecewise constant (intervals of size h) versions of the Markov Chain are compact in $D[[0,T],R]$. We impose conditions on the behavior of

$$a_h(x)=\frac{1}{h}\int_{|y-x|\leq\delta}(y-x)^2\pi_h(x,dy)$$

and

$$b_h(x)=\frac{1}{h}\int_{|y-x|\leq\delta}(y-x)\pi_h(x,dy)$$

to determine any limit. We assume that

$$\sup_{\substack{0<h\le 1\\ 0\le x\le \ell}} a_h(x) \le C(\ell)$$

$$\inf_{\substack{0<h\le 1\\ 0\le x\le \ell}} b_h(x) \ge -C(\ell)$$

and uniformly on compact subsets of $(0,\infty)$,

$$\lim_{h\to 0} a_h(x) = 1$$

$$\lim_{h\to 0} b_h(x) = 0$$

For smooth functions f with compact support in $(0,\infty)$, by a Taylor expansion

$$\lim_{h\to 0}\frac{1}{h}\int [f(y)-f(x)]\pi_h(x,dy) \to \frac{1}{2}f''(x)$$

locally uniformly. Any limit is therefore Brownian motion away from 0. We need only determine what happens at 0. The parameter γ is to be determined from the behavior of $a_h(x), b_h(x)$ as $x, h \to 0$.

Theorem 16.4.

1. *If*

$$\liminf_{\substack{h\to 0\\ x\to 0}} a_h(x) \ge c > 0$$

then the limit exists and is the reflected Brownian motion, i.e. $\gamma = 0$.

2. *If* $b_h(x) \to +\infty$ *whenever* $x, h, a_h(x) \to 0$ *then again the limit is the reflected Brownian motion.*

3. *If* $b_h(x)$ *remains uniformly bounded as* $h, x \to 0$ *and tends to* γ *whenever* $h \to, 0, x \to 0$ *and* $a_h(x) \to 0$*, then the limit exists and is the process with generator*

$$(\mathcal{L}_\gamma f)(x) = \frac{1}{2}\mathbf{1}_{(0,\infty)}(x)f''(x) + \gamma \mathbf{1}_{\{0\}}(x)f'(0)$$

Proof. For proving **1** or **2**, we need to show that for any limit P of $P_{h,x}$ as $h, x \to 0$, for some $\ell > 0$,

$$E^P[\int_0^{\tau_\ell\wedge t} \mathbf{1}_{\{0\}}(x(s))ds] = 0$$

where τ_ℓ is the exit time from $[0,\ell]$. Take $\ell = 1$. By Taylor expansion we see that

$$\frac{1}{h}\int [f(y)-f(x)]\pi_h(x,dy) = b_h(x)f'(x) + \frac{1}{2}a_h(x)f''(x) + \Delta(h)$$

In particular if we consider $f = f_\epsilon(x)$ defined in (16.5), it is uniformly bounded by 2ϵ on $[0,1]$. and

$$\int [f(y)-f(x)]\pi_h(x,dy) = h[b_h(x)f'(x)+\frac{1}{2}a_h(x)f''(x)]+o(h) = (L_j f)(x)+o(h)$$

uniformly on $[0,1]$.

$$(L_h f)(x) = [b_h(x)f'(x) + \frac{1}{2}a_h(x)f''(x)] \geq c\mathbf{1}_{[0,\epsilon]}(x) - 2C(1)\epsilon$$

$$f(X(nh)) - f(X(0)) - \sum_{j=0}^{(n-1)} \int [f(y) - f(X(jh))]\pi(X(jh), dy)$$

is a martingale. Therefore we obtain in the limit

$$f(x(t\wedge\tau_1)) - f(x(0)) - \int_0^{t\wedge\tau_1} c\mathbf{1}_{[0,\epsilon]}(x) - 2C(1)\epsilon$$

is a sub-martingale. This provides an estimate

$$E^P[\int_0^{t\wedge\tau_1} \mathbf{1}_{[0,\epsilon]}(x(s))ds] \leq K\epsilon(t+1)$$

which proves **1**. To prove **2** we need a different test function ϕ. Take $\phi(x) = f(x) + \delta x$ on $[0,1]$. Then on $[0,1]$, ϕ and ϕ' are bounded by $C(\epsilon+\delta)$ and for some $L_\theta \to \infty$ as $\theta \to 0$,

$$\liminf_{h\to 0}[b_h(x)\phi'(x) + \frac{1}{2}a_h(x)\phi''(x)] \geq (L_\theta\delta\wedge\theta)\mathbf{1}_{[0,\epsilon]}(x) - C(\epsilon+\delta)$$

This provides the estimate for the limiting P

$$E^P[\int_0^{t\wedge\tau_1} [L_\theta\delta\wedge\theta]\mathbf{1}_{[0,\epsilon]}(x(s))ds] \leq C(\epsilon+\delta)$$

and this is enough. Let $\epsilon \to 0$ to get

$$\limsup_{\epsilon\to 0} E^P[\int_0^{t\wedge\tau_1} \mathbf{1}_{[0,\epsilon]}(x(s))ds] \leq C\frac{\delta}{L_\theta\delta\wedge\theta}$$

and now pick $\theta = \delta$ and let $\delta \to 0$. Finally we turn to **3**. We need to consider for smooth $u(s,x)$, with $u_s(s,0) + \gamma u_x(s,0) \geq 0$,

$$\int [u(s+h,y) - u(s,x)]\pi_h(x,dy)$$

and determine its sign when x is close to 0 and h is small. We see that this is nearly

$$h[u_s(s,x) + b_h(x)u_x(s,x) + \frac{1}{2}a_h(x)u_{xx}(s,x)]$$

From earlier estimate we know that the chain will not spend much time close to 0 where $a_h(x)$ is not small and then $b_h(x)$ is close to γ. Since $u_s(s,0)+\gamma u_x(s,0) \geq 0$ this contribution is nonnegative. □