Differential equations

A differential equation is a relationship between a function and its derivatives.

Examples: $f(x)$ is the function $f'(x), f''(x), \cdots$ are the derivatives. A differential equation is a relation of the form $\phi(f, f', f'', \cdots) = 0$.

1. $f'(x) = \sin x$
2. $f'(x) = f(x)\sin x$
3. $f''(x) = 1$

Solutions:

1. $f(x) = -\cos x + c$
2. $f(x) = ce^{-\cos x}$
3. $f(x) = \frac{x^2}{2} + c_1 x + c_2$

The order of the equation is the order of the highest derivative involved. It is $1, 2$ and 2 respectively in our examples. Solutions in general have a number of free constants equal to the order of the equation.

We will concentrate on first order equations that involve $x, y = f(x)$ and $\frac{dy}{dx} = f'(x)$. We can rewrite them in the form

$$\frac{dy}{dx} = g(x, y)$$

In general one can not solve this explicitly. But there are some cases when we can.

$\frac{dy}{dx} = g(x)$. This is just integrating g. $y(x) = \int^x g(x)dx + c = G(x) + c$.

$\frac{dy}{dx} = g(y)$. We rewrite this as $\frac{dx}{dy} = g(y)$. $x = \int^y g(y)dy + c = G(y) + c$ We can also have $\frac{dy}{dx} = g(x)h(y)$. We rewrite as

$$\frac{dy}{h(y)} = g(x)dx$$

Integrating we get

$$\int^y \frac{1}{h(y)} dy = \int^x g(x)dx + c$$

which we can then solve for y. For example

$$\frac{dy}{dx} = y\sin x, \quad \frac{dy}{y} = \sin x\, dx, \quad \log y = -\cos x + c, \quad y = e^{-\cos x + c} = c\, e^{-\cos x}$$

$$\frac{dy}{dx} = \sqrt{1 - y^2} e^x$$

$$\frac{dy}{\sqrt{1 - y^2}} == e^x dx$$

$$\arcsin y = e^x + c$$
$$y = \sin(e^x + c)$$

First order linear equations: $\frac{dy}{dx} = p(x)y + q(x)$. Let $P(x)$ be the integral of $p(x)$ i.e $P'(x) = p(x)$. We can rewrite the equation as

$$\frac{dy}{dx}e^{-P(x)} - ye^{-P(x)}p(x) = e^{-P(x)}q(x)$$
$$\frac{d[ye^{-P(x)}]}{dx} = e^{-P(x)}q(x)$$
$$ye^{-P(x)} = \int^x e^{-P(x)}q(x)dx + c$$
$$y = e^{P(x)}\int^x e^{-P(x)}q(x)dx + c\,e^{P(x)}$$

Example: Consider a tank in which inflow is at a constant rate of c units per second and outflow is at rate which is proportional to the current storage level. If $x = x(t)$ is the amount stored at time t the inflow in a small time dt is $c\,dt$ and the outflow is $rx(t)dt$ where is the constant of proportionality. This can be summarized as

$$\frac{dx(t)}{dt} = -rx(t) + a$$

The solution is obtained as

$$\frac{dx(t)}{dt} + rx(t) = a; \quad \frac{d[e^{rt}x(t)]}{dt} = ae^{rt}; \quad e^{rt}x(t) = c + \frac{a}{r}e^{rt}; \quad x(t) = ce^{-rt} + \frac{a}{r}$$

The value of c can be determined if we know $x(0)$, the storage level at time 0.

Example: Consider a bank account where deposits are made (continuously) at a rate of a units of currency per year and the interest rate on current balance is r per year, compounded continuously. Then the balance $x(t)$ after time t satisfies

$$\frac{dx(t)}{dt} = rx(t) + a$$

The solution then is

$$x(t) = ce^{rt} - \frac{a}{r}; \quad x(0) = c - \frac{a}{r}; \quad c = x(0) + \frac{a}{r}$$

If $x(0) < 0$ you have a loan, that you are paying off.

$$x(t) = [x(0) + \frac{a}{r}]e^{rt} - \frac{a}{r}$$

If $a < -rx(0)$ your payments do not even cover the current interest and you are getting deeper in the hole. If $x(0) + \frac{a}{r} > 0$ then solving for $x(t) = 0$

$$t = \frac{1}{r}\log\frac{a}{a + rx(0)}$$

is when your loan is finally paid off.