Model Questions.

Definite and Indefinite Integration

1. $\int_0^1 \frac{x}{\sqrt{1-x}} dx$

2. $\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x}} dx$

3. $\int_0^1 x\, e^x dx$

4. $\int x \,\sin x \,dx$

5. $\int \sin^2 2x \,dx$

6. $\int e^x \sin x dx$

7. $\int_0^\infty \frac{dx}{100+x^2}$

8. Why is $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$ for all f?

9. Can you use it to show $\int_0^{\frac{\pi}{2}} (\sin x)^2 dx = \int_0^{\frac{\pi}{2}} (\cos x)^2 dx = \frac{\pi}{4}$ with out explicit calculation.

10. $\int_0^1 e^{sinx} \cos x dx$

Convergence of Integrals and series. See if the following integrals or series converge or do not

1. $\int_0^\infty e^{x-x^2} dx$

2. $\int_1^\infty \frac{e^x}{1+e^{2x}} dx$

3. $\int_0^1 \frac{1}{1-\cos x} dx$

4. $\int_0^1 \frac{1}{1-\cos \pi x} dx$

5. For what values of p will $\int_0^1 \frac{1-\cos x}{x^p} dx$ converge?

6. $\sum_n e^n$

7. $\sum_n \frac{e^n}{n!}$

8. $\sum_n \frac{(-1)^n}{\log(n+1)}$

9. $\sum_n n^{100} e^{-\sqrt{n}}$

10. $\sum(-1)^n$

Evaluate the following limits, if it exists.

1. $\lim_{n\to\infty} \frac{n^2}{2n^2+7}$

2. $\lim_{n\to\infty} \frac{n^{100}}{e^{\sqrt{n}}}$

3. $\lim_{n\to\infty} n \sin \frac{1}{n}$

4. $\lim_{n\to\infty} n \cos \frac{1}{n}$

5. $\lim_{n\to\infty} n \tan \frac{1}{n}$

6. $\lim_{n\to\infty} [\frac{n+1}{n}]^n$

7. $\lim_{n\to\infty} \frac{\log n}{n}$

8. $\lim_{n\to\infty} n\, e^{-n}$

9. $\lim_{n\to\infty} n^4(1 - \cos \frac{1}{n})^2$

10. $\lim_{n\to\infty} [1 + e^{-1} + e^{-2} + \cdots + e^{-n}]$

Answers (Hints)

1. Put $x = 1 - y^2$ or $x = \sin^2 \theta$.

2. Put $x = \sin^2 \theta$

3. Integrate by parts. $\int xe^x dx = \int xde^x = xe^x - \int e^x dx = xe^x - e^x + c$

4. Integrate by parts. $\int x \sin x \, dx = -\int x \, d\cos x = -x\cos x + \int \cos x dx = -x \cos x + \sin x + c$

5.$sin^2 2x = \frac{1-\cos 4x}{2}$. Therefore $\int \sin^2 2x = \frac{x}{2} - \frac{\sin 4x}{8} + c$

6.$\int e^x \sin x dx = \int \sin x de^x = \sin x e^x - \int e^x \cos x dx = \sin x e^x - \int \cos x de^x = \sin x e^x - \cos x e^x - \int \sin x e^x dx$. Therefore $\int \sin x e^x dx = \frac{1}{2} \sin x e^x - \cos x e^x + c$

7. $\int_0^\infty \frac{dx}{100+x^2}$. Put $x = 10y$. $\int_0^\infty \frac{dx}{100+x^2} = \int_0^\infty \frac{10dy}{100+100y^2} = \frac{1}{10}\frac{\pi}{2}$

8. Put $y = \frac{\pi}{2} - x$.

9. The sum is $\int_0^{\frac{\pi}{2}} 1dx = \frac{\pi}{2}$.

10. Put $y = \sin x$. $e^{\sin 1} - 1$.

1. x^2 is a lot bigger than x. $x^2 \geq 2x - 1$ will do it. Converges

2.$\frac{e^x}{1+e^{2x}} \leq e^{-x}$ Converges.

3. $1 - \cos x \simeq \frac{x^2}{2}$ near 0. No chance.

4. Makes no diffrence. Same as 3. $1 - \cos \pi x \simeq \frac{\pi^2 x^2}{2}$

5. $p < 3$. same as $\int_0^1 \frac{x^2}{x^p} dx$.

6. No chance. e^n grows.

7. Ratio test. $\frac{a_{n+1}}{a_n} \to 0$. Converges.

8. Alternating monotone. $\frac{1}{\log(n+1)} \to 0$. Convereges

9 9. $e^{-\sqrt{n}} n^{100}$ is very small. Smaller than $\frac{1}{n^2}$ Show $n^{102} e^{-\sqrt{n}} \to 0$.

10. The terms do not go to 0. No chance.

Limits.

1. $\frac{1}{2}$.

2. 0

3. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ by L'Hospital.

4. ∞ No limit.

5. Same as 3. 1.

6. Take logs. $n \log \frac{n+1}{n} = n \log(1 + \frac{1}{n}) = \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ So the limit is $e^1 = e$.

7. 0 L'Hospital again with $x = \frac{1}{n}$

8. 0

9. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$. Therefore

$$\lim_{n\to\infty} n^4 (1 - \cos \frac{1}{n})^2 = \lim_{x\to 0} \frac{(1-\cos x)^2}{x^4} = \frac{1}{4}$$

10. $s_n = \frac{1-e^{-(n+1)}}{1-e^{-1}}$. $\lim_{n\to\infty} s_n = 1 + e^{-1} + \cdots + e^{-n} + \cdots = \frac{1}{1-e^{-1}} = \frac{e}{e-1}$