

**Oct 12, 2009.**

Approximating functions. If  $f(x)$  is a complicated function of  $x$  can we approximate it by a simpler function. The simpler functions are usually polynomials of low degree. Constants, linear expression and quadratic expressions. You can use the approximations to evaluate integrals. For instance to calculate  $\int_0^1 \frac{1}{1+x^3} dx$ . The function  $f(x)$  has values

$$f(0) = 1, f\left(\frac{1}{4}\right) = \frac{64}{65}, f\left(\frac{1}{2}\right) = \frac{8}{9}, f\left(\frac{3}{4}\right) = \frac{64}{91}, f(1) = \frac{1}{2}$$

Can we use this to evaluate the integral approximately?

Trapezoidal Rule; Approximate the function by a linear function in each of the intervals  $[0, \frac{1}{4}]$ ,  $[\frac{1}{4}, \frac{1}{2}]$ ,  $[\frac{1}{2}, \frac{3}{4}]$ ,  $[\frac{3}{4}, 1]$ .

If  $(a, f(a)), (b, f(b))$  are two points the equation of the line joining them is

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

or

$$y = f(a) + (x - a) \frac{f(b) - f(a)}{b - a}$$

and

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^b [f(a) + (x - a) \frac{f(b) - f(a)}{b - a}] dx \\ &= (b - a)f(a) + \frac{f(b) - f(a)}{b - a} \int_a^b (x - a) dx \\ &= (b - a)f(a) + \frac{f(b) - f(a)}{b - a} \left[ \frac{(x - a)^2}{2} \right]_a^b \\ &= (b - a)f(a) + \frac{f(b) - f(a)}{b - a} \frac{(b - a)^2}{2} \\ &= (b - a)[f(a) + \frac{f(b) - f(a)}{2}] \\ &= (b - a) \frac{[f(a) + f(b)]}{2} \end{aligned}$$

In our problem then

$$\begin{aligned} I &= \frac{1}{4} \left[ \frac{f(0) + f\left(\frac{1}{4}\right)}{2} + \frac{f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right)}{2} + \frac{f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)}{2} + \frac{f\left(\frac{3}{4}\right) + f(1)}{2} \right] \\ &= \frac{1}{8} [f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)] \\ &= \frac{1}{8} \left[ 1 + 2\left(\frac{64}{65} + \frac{8}{9} + \frac{64}{91}\right) + \frac{1}{2} \right] \\ &\simeq 0.8317 \end{aligned}$$

Fit a Quadratic. Simpson's Rule.

Parabola through  $[-h, f(-h)), (0, f(0)), (h, f(h))]$ .

$$f(x) = ax^2 + bx + c$$

$$c = f(0); ah^2 + bh + c = f(h), ah^2 - bh + c = f(-h)$$

Solve for  $b$  and  $c$ . Simultaneous equations, two unknowns.

$$a = \frac{f(h) - 2f(0) + f(-h)}{2h^2}, b = \frac{f(h) - f(-h)}{4h}$$

$$\begin{aligned} I &= \int_{-h}^h [ax^2 + bx + c] dx = \frac{2ah^3}{3} + 2ch \\ &= \frac{h}{3}[f(-h) - 2f(0) + f(h)] + 2hf(0) \\ &= \frac{h}{3}[f(-h) + 4f(0) + f(h)] \end{aligned}$$

In our example

$$\begin{aligned} \frac{1}{12}[f(0) + 4f(\frac{1}{4}) + 2f(\frac{1}{2}) + 4f(\frac{3}{4}) + f(1)] &= \frac{1}{12}[1 + 4\frac{64}{65} + 2\frac{8}{9} + 4\frac{64}{91} + \frac{1}{2}] \\ &\simeq 0.8358 \end{aligned}$$

Actual value?

$$\begin{aligned} \int \frac{dx}{1+x^3} &= \frac{1}{3} \log(1+x) - \frac{1}{6} \log(1-x+x^2) + \frac{1}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c \\ &= \frac{1}{3} \log 2 + 0 + \frac{1}{\sqrt{3}} \left[ \arctan \frac{1}{\sqrt{3}} - \arctan \frac{-1}{\sqrt{3}} \right] \\ &= \frac{1}{3} \log 2 + \frac{\pi}{3\sqrt{3}} = 0.8364 \end{aligned}$$

**Homework.** Calculate the following integrals from 0 to 1 by taking the values at points spaced 0.1 apart by using trapezoidal rule and Simpson's rule. Compare the two values with the actual values by integrating, using calculator or tables as necessary.

$$1. \int_0^1 (2x+3)dx$$

$$2. \int_0^1 (1+x^2)dx$$

$$3. \int_0^1 (\sin 10\pi x)^2 dx$$

$$4. \int e^{-x} dx$$