October 5, 09

There are a lot of similarities between convergence of improper integrals and infinite series.

Let us consider only integrals of the form $\int_a^\infty f(x)dx$ with $f(x) \geq 0$ and $\sum_{n=1}^\infty a_n$ with $a_n \geq 0$.

Some examples of integrals that converge are

1. $\int_0^\infty e^{-ax}dx$ with $a > 0$
2. $\int_1^\infty \frac{dx}{x^p}$ with $p > 1$.

Other integrals are often compared to these to prove convergence. Examples of integrals that diverge are

$\int_1^\infty \frac{dx}{x^p}$ with $p \leq 1$. Integrals are again compared with these to show divergence. That is if $g(x) \geq \frac{c}{x^p}$ with $p \leq 1$ for sufficiently large x then $\int_1^\infty g(x)dx = \infty$.

Similarly for series. Examples of convergent series are

1. $\sum_{n=0}^\infty \rho^n$ for $0 \leq \rho < 1$, the geometric series.
2. $\sum n = 1^\infty \frac{1}{n^p}$ for $p > 1$

Other series are compared to these. If a_n for large n is bounded by a constant multiple of any of these it converges. If it is greater than a positive constant multiple of $\frac{1}{n^p}$ with $p \leq 1$ then it diverges.

Integrals and series can be compared to each other. If $f(x)$ is monotone decreasing then

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n)$$

Adding up

$$\sum_{n=2}^\infty f(n) \leq \int_1^\infty f(x)dx \leq \sum_{n=1}^\infty f(n)$$

They both behave the same way. Monotonicity is only needed for sufficiently large x.

Ways of comparing to ρ^n or n^p.

Ratio Test: If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$ then if we pick a value r, such that $\rho < r < 1$, then $a_n \le Cr^n$ for large n and since $r < 1$ we have convergence. Similarly if $\rho > 1$ we can compare $a_n \ge cr^n$ for some r with $\rho > r > 1$ and we will have divergence. This test is inconclusive if $\rho = 1$.

Root Test. If $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \rho$ and $\rho < 1$ we can pick a value r, such that $\rho < r < 1$. Then $a_n \le r^n$ for large n and since $r < 1$ we have convergence. Similarly if $\rho > 1$ we will have $a_n \ge r^n$ for some r with $\rho > r > 1$ and we will have divergence. This test is inconclusive if $\rho = 1$.

Comparing to $\frac{1}{n^p}$ is harder.

$$\frac{a_{n+1}}{a_n} = \frac{n^p}{(n+1)^p} = (\frac{n}{n+1})^p \to \rho = 1$$

We can not distinguishthe different values of p. Calculate

$$\lim_{n\to\infty} n\left[1 - (\frac{n}{n+1})^p\right] = \lim_{n\to\infty} n\left[1 - (\frac{1}{1+\frac{1}{n}})^p\right] = \lim_{x\to 0} \frac{1-(1+x)^{-p}}{x} = p$$

by L'Hospital's rule. This would suggest that if

$$\lim_{n\to\infty} n\left[1 - (\frac{a_{n+1}}{a_n})\right] = p$$

then $p > 1$ would imlply convergence and $p < 1$ divergence.

Homework. Check if the following series converge or diverge.

$\sum_n \frac{(10)^n}{n^n}$, $\quad \sum_n \frac{1}{\sqrt{n!}}$, $\quad \sum_n \frac{1}{\binom{2n}{n}}$. $\quad \sum_n \frac{1}{2^{n^2}}$. $\quad \sum_n e^{-(\log n)^2}$.