September 30, 09

The Quiz this time was a disappointment. We will go over it and do more problems on improper integrals.

1. $\int_0^\infty \frac{e^x}{1+e^x}dx$. Diverges

How to proceed? The function $f(x) = \frac{e^x}{1+e^x}$ is bounded by 1. In fact if $x \geq 0$, $\frac{1}{2} \leq f(x) \leq 1$ and $f(x) \to 1$ as $x \to \infty$. The trouble comes only from ∞. You can say

$$f(x) \geq \frac{1}{2}$$

and so $\int_0^\infty f(x)dx \geq \int_0^\infty \frac{1}{2}dx = \infty$ or try to integrate it

$$\lim_{x\to\infty} \int_0^x \frac{1+e^y}{e^y}dy = \lim_{x\to\infty} \log[1+e^x] = \infty$$

2. $\int_0^\infty \frac{e^x}{1+e^{2x}}dx$. Converges

Change variables. $u = e^x$. The integral becomes $\int \frac{1}{1+u^2}du$. We need to evaluate

$$\int_1^\infty \frac{du}{1+u^2} = \arctan u|_1^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Or use comparison $\frac{e^x}{1+e^{2x}} \leq \frac{e^x}{e^{2x}} = e^{-x}$ and $\int_0^\infty e^{-x}dx = -e^{-x}|_0^\infty = 1$

3. $\int_0^1 \frac{(\sin x)^2}{x^{\frac{5}{2}}}dx$. Converges

This too complicated to integrate. Need to estimate. Where is the trouble? Only when $x = 0$. When $x \simeq 0$, $\sin x \simeq x$ so we are looking at $f(x) =\simeq \frac{x^2}{x^{\frac{5}{2}}} = x^{-\frac{1}{2}}$ Since $\frac{1}{2} < 1$ the integral converges. Remember that $\int_0^1 \frac{dx}{x^p}$ converges if and only if $p < 1$.

4. $\int_0^1 \frac{1}{\sin x}dx$. Diverges

Same idea. Now $\frac{1}{\sin x} \simeq \frac{1}{x}$ and now $p = 1$ and $\int_0^1 \frac{dx}{x^p}$ diverges at 0.

5. $\lim_{n\to\infty} \frac{1}{\log\log n} = 0$

If n is big so is $m = \log n$. If m is big so is $\log m = \log\log n$. Therefore $\frac{1}{\log\log n}$ is small and tends to 0.

6.$\lim_{n\to\infty} n^2(1 - \cos\frac{1}{n}) = \frac{1}{2}$.

This takes the form $\infty \cdot (1-\cos 0) = \infty \cdot 0 = ?$. Rewrite with $x = \frac{1}{n}$ and apply L'Hospital's rule twice.

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim x \to 0 \frac{\cos x}{2} = \frac{1}{2}.$$

7.$\lim_{n\to\infty} e^{e^{-n}} = 1$.

This is the easy one. $e^{-n} = \frac{1}{e^n} \to 0$ as $n \to \infty$ and $e^0 = 1$.

8.$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

This is of the form ∞^0 and needs L' Hospital's rule. Taking logarithms

$$\log n^{\frac{1}{n}} = \frac{\log n}{n}$$

and

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

so that $\lim_{n\to\infty} a_n = e^0 = 1$

Homework.

Do the following integrals converge or diverge?

1. $\int_0^1 \frac{dx}{\sqrt{x(1-x^5)}}$

2 $\int_1^\infty [\log \frac{(x+1}{x}]^2 dx$

3. $\int_1^\infty \frac{1}{1+x}^p dx$

4. $\int_0^\infty e^{-x} x^{-\frac{1}{2}} dx$

5. $\int_0^{\frac{\pi}{2}} \tan x dx$