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An infinite series is an attempt to add an infinite number of terms. For example

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots\cdots = 2$$

It is interpreted in the following manner. We sum the first n terms.

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
S_3 &= 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} \\
S_4 &= \frac{15}{8} \\
&\cdots\cdots \\
S_n &= \frac{2^n - 1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}
\end{aligned}$$

Then we take the limit as $n \to \infty$. In this case the limit exists and equals 2. We then say that the infinite series **converges** and the sum of the series is 2. Not all series converge.

$$1 + 1 + 1 + 1 + \cdots$$

does not converge. $S_1 = 1, S_2 = 2, \ldots, S_n = n$ and S_n does not converge. It diverges to $+\infty$. Series can oscillate too.

$$1 - 1 + 1 - 1 + \cdots \pm 1 \pm$$

$$S_1 = 1, S_2 = 0, S_3 = 1, S_4 = 0, \ldots$$

The odd terms $S_{2n+1} = 1$ and the even terms $S_n = 0$. S_n does not converge. For S_n tp oscillate the individual terms a_n must change sign, i.e be both positive and negative. A series of positive terms will have partial sums S_n that will form an **increasing** sequence. If bounded it will converge to a finite limit. Otherwise it will diverge to $+\infty$. A series of negative terms is no different. The partial sums now decrease and if bounded below will converge to a limit. Otherwise will diverge to $-\infty$. A series with both positive and negative terms may converge, diverge to $\pm\infty$ or oscillate with no limit.

Examples.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} + \cdots$$

Note that $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. Therefore

$$S_n = [1 - \frac{1}{2}] + [\frac{1}{2} - \frac{1}{3}] + \cdots [\frac{1}{n} - \frac{1}{n+1}] = 1 - \frac{1}{n+1} \to 1$$

What about

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots ?$$

Note that

$$\frac{1}{n^2} \leq \frac{1}{(n-1)n}$$

Therefore

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots \leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} + \cdots = 1 + 1 = 2$$

The series converges. This leads to the comparison test. If $a_n, b_n \geq 0$ and $a_n \leq C b_n$ and $\sum_n b_n$ converges then so does $\sum_n a_n$.

Examples. The series $\sum_n \frac{1}{n}$ diverges.

$$\frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} \geq (2^{n+1} - 2^n) \frac{1}{2^{n+1}} = \frac{2^n}{2^{n+1}} = \frac{1}{2}$$

Therefore

$$\sum_n \frac{1}{n} \geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} + \cdots = \infty$$

Comparing with integrals. Consider $\sum_n \frac{1}{n^p}$. Let $p > 1$.

$$\frac{1}{n^p} \leq \int_{n-1}^{n} \frac{dx}{x^p} = \frac{1}{p-1}[(n-1)^{1-p} - n^{1-p}$$

because $\frac{1}{n^p}$ is the smallest value of $\frac{1}{x^p}$ in the interval $[n-1, n]$. Summing over n, we have

$$\sum_{n=2}^{N} \frac{1}{n^p} \leq \int_1^N \frac{dx}{x^p} = \frac{1}{p-1}[1 - \frac{1}{N^{p-1}}]$$

We can always compare a sum to an integral. If $f(x) \geq 0$ is a monotone decreasing function, then $\sum_n f(n)$ converges if and only if $\int_1^\infty f(x)dx$ converges. Just note

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n)$$

Therefore

$$\sum_{n=2}^{\infty} f(n) \leq \int_1^\infty f(x)dx \leq \sum_{n=1}^{\infty} f(n)$$

Homework. Examine the following series and see if it converges or diverges.

1. $\sum_{n=3}^{\infty} \frac{1}{n \log n}$
2. $\sum_{n=1}^{\infty} e^{-n}$
3. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$
4. $\sum_{n=1}^{\infty} \frac{n}{2^n}$
5. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^4}$