September 23, 2009.

Sequences, limits and series (continued).

If $f(x)$ is function defined on $[a, \infty)$ i.e for all $x \geq a$ we can talk about

$$\lim_{x\to\infty} f(x) = \ell$$

This means given any accuracy, i.e, a small number ϵ

$$|f(x) - \ell| \leq \epsilon$$

for all sufficiently large x, i.e. for $x \geq A$. One can some times use L'Hospital's rule to determine the limit. If $f(x) \to \ell$ as $\ell \to \infty$ and $a_n = f(n)$, then $a_n \to \ell$ as $n \to \infty$. This is obvious because, if some thing is true for all $x \geq A$, it is of course true for all integers larger than A. The converse is not always true. $f(x) = \sin 2x\pi$ is a periodic function and has no limit. But if you look at it points of the form $y + 2n\pi$, then

$$a_n = \sin(y + 2n\pi) = \sin y \to \sin y$$

You can use this to calculate some limits.

Example:

$$a_n = n^k e^{-cn}$$

for any power k and $c > 0$. We look at function

$$f(x) = x^k e^{-c\,x}$$

for $x \geq 1$. We write it as

$$f(x) = \frac{x^k}{e^{c\,x}}$$

If $k > 0, c > 0$ this is of the form $\frac{\infty}{\infty}$. If $k \leq 0, c > 0$ then the numerator is not big and the denominator gets big for large x and the limit is 0. When $k > 0$ apply L'Hospital's rule. Then we reduce the problem to

$$\frac{k}{c} \lim_{x\to\infty} \frac{x^{k-1}}{e^{c\,x}}$$

i.e k become $k - 1$. The power reduces by 1. If we keep repeating the rule eventually k becomes 0 if we start from an integer value or less than 0 otherwise. In any case

$$\lim_{x\to\infty} x^k\, e^{-c\,x} = 0$$

and therefore

$$\lim_{n\to\infty} n^k\, e^{-c\,n} = 0$$

In other words e^{cn} is much bigger than n^k for any k. We write $e^{cn} >> n^k$.

Example: Ratio of polynomials.

$$a_n = \frac{a_0 n^k + a_1 n^{k-1} + \cdots + a_k}{b_0 n^p + b_1 n^{p-1} + \cdots + b_p}$$

with a_0, b_0 not equal to 0. Look at

$$f(x) = \frac{a_0 x^k + a_1 x^{k-1} + \cdots + a_k}{b_0 x^p + b_1 x^{p-1} + \cdots + b_p}$$

There are three cases. $k < p, k = p, k > p$. In the first case, after applying the rule k times, we end up with

$$\frac{a_0(k!)}{b_0[p(p-1)\cdots(p-k+1)]x^{p-k} + \cdots}$$

which tends to 0. In the second case we end up with

$$\frac{a_0(k!)}{b_0(k!)} = \frac{a_0}{b_0}$$

which is the limit. In the third case we end up with

$$\frac{a_0[k(k-1)\cdots(k-p+1)]x^{k-p} + \cdots}{b_0(p!)}$$

which tends to $\pm\infty$ depending on the sign of $\frac{a_0}{b_0}$.

Example: If f is continuous at $x = \ell$ and $a_n \to \ell$ then $f(a_n) \to f(\ell)$. This can be used; for instance

$$\lim_{n\to\infty} \sqrt{1 + e^{-n}} = 1$$

A sequence a_n is bounded above if there is a number M such that

$$a_n \leq M$$

for all n.

$$\lim_{n\to\infty} \sqrt{1 + e^{-n}} = 1$$

and a sequence a_n is bounded below if there is a number m such that

$$a_n \geq m$$

for all n. A bounded sequence is one that is bounded above and below. We can take the bounds to be $\pm M$ so that we have

$$|a_n| \leq M$$

for all n. A convergent sequence is always bounded, but the converse is not true. If a sequence is convergent to a limit ℓ then

$$|a_n - \ell| \leq 1$$

for $n \geq k$ some k. Then

$$M = \max\{|a_1|, \ldots, |a_k|, |\ell| + 1\}$$

will work. A monotone increasing sequence is bounded below and will converge to a limit if it is bounded above. The limit is actually the least upper bound, defined as the smallest number which is still an upper bound.

Example:

$$a_n = \arctan n$$

$$f(x) = \arctan x; \quad f'(x) = \frac{1}{1+x^2} \geq 0$$

So $\arctan x$ is increasing. It is also bounded by $\frac{\pi}{2}$. In fact the limit is $\frac{\pi}{2}$.

Home Work.

Determine for each of the following sequences if it converges, is bounded but does not converge or unbounded.

1. $a_n = n \sin n$
2. $a_n = \sin(n^2 + 1)$
3. $a_n = \frac{n}{n+1}$
4. $a_n = (-2)^n$
5. $a_n = 11(-1)^n$