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Sequences, limits and series.

A sequence is a function defined on the set of positive integers. For $n = 1, 2, 3, \ldots$, the value a_n of the function has to be given.

Examples:

Ex 1. $\{1, 1, 1, 1, 1, \ldots\}$. $a_n = 1$ for all n.

Ex 2. $\{1, 2, 3, 4, 5, \ldots\}$. $a_n = n$

Ex 3. $\{1, -1, 1, -1, 1, -1, \ldots\}$. $a_n = (-1)^{n-1}$.

Ex 4. $\{0.1, 0.01, 0.001, 0.0001, \ldots.\}$ $a_n = (10)^{-n}$.

A sequence does not have start from $n = 1$. Can start from $n = 0$ or $n = 5$ or from any value of n. For example $a_n = \sqrt{n-7}$ is defined for $n \geq 7$. But it has to be defined for all $n \geq k$ from some k. It is usually expressed by means of a formula. $a_n = \sin n$, $a_n = n^{\frac{1}{3}}$, $a_n = \frac{n^2-1}{n^2+1}$ are all sequences. But it could be defined by other means as well. For example a_n can be the largest integer not exceeding $\sqrt{n}$.

$$a_1 = a_2 = a_3 = 1, a_4 = a_5 = a_6 = a_7 = a_8 = 2, a_9 = \cdots = a_{15} = 3, \cdots\cdots$$

You can not tell what the sequence is if you just know the first few terms. For instance a question like what is the next member of the sequence

$$1, 4, 9, 16, 25, \ldots$$

DOES NOT MAKE SENSE. You can guess, but there is no guarantee and who ever generated the sequence can change his or her mind.

If we are interested in the behavior of the sequence for large n, i.e as $n \to \infty$ then the first few terms do not matter.

Limits of Sequences. A sequence $\{a_n\}$, is said to have a limit a as $n \to \infty$ if a_n is close to a for large n. It may not be close in the beginning, but eventually it is close and stays close. The emphasis here is on the words "close" and "eventually". How close is close ? And how long do I have to wait before a_n to be that close. We can specify the accuracy say 10^{-4} and we would then demand that a_n should be in the interval $(a - 10^{-4}, a + 10^{-4})$ from certain point on. If we increase the accuracy and demand a_n belong to say $(a - 10^{-8}, a + 10^{-8})$ it should still happen after a certain point, but the wait may be longer. For any degree of accuracy it happens eventually that a_n is close to a with that degree of accuracy, but the wait gets longer as the needed accuracy gets more demanding.

Let us illustrate this by a simple example. $a_n = \frac{1}{n}$. It is clear that for large n, $\frac{1}{n}$ is small and therefore $\frac{1}{n}$ should tend to the limit 0, i.e a in this case is 0. If we want $\frac{1}{n}$ to be in the interval $(-10^{-4}, 10^4)$ that will happen if $n > 10^4$ and $\frac{1}{n}$ will be in the interval $(-10^{-8}, 10^8)$ if $n > 10^8$ and so on.

Note that a_n tending to a limit a is the same as $b_n = a_n - a$ tending to the limit 0. a_n is in $(a - 10^{-4}, a + 10^{-4})$ is the same as $a_n - a$ is in $(-10^{-4}, 10^{-4})$.

An easy but useful fact is if $|b_n| \leq C|a_n|$ and $a_n \to 0$ as $n \to \infty$ (i.e a_n has limit zero) then b_n tends to 0 as well. C is a constant say $|C| \leq 10^6$. Then if we want an accuracy of 10^{-8} i.e. need $|b_n| \leq 10^{-8}$, just make $|a_n| \leq 10^{-14}$.

1.

$$|\frac{\sin n}{n}| \leq \frac{1}{n}; \quad \frac{\sin n}{n} \to 0$$

2.

$$|\frac{n}{n^2+1}| = \frac{n}{n^2+1} \leq \frac{n}{n^2} \leq \frac{1}{n}; \quad \frac{n}{n^2+1} \to 0$$

3.

$$a_n = \frac{n^2 - 6n + 7}{n^3 - n^2 - n + 3}$$

It is $\frac{n^2}{n^3} = \frac{1}{n}$ that matters. The numerator

$$|n^2 - 6n + 7| \leq n^2 + 6n + 7 \leq n^2 + 13n \leq 14n^2 \qquad \text{if } n \geq 1$$

The denominator

$$n^3 - n^2 - n + 3 \geq n^3 - 2n^2 = n^2(n-2)$$

For $n \geq 3$

$$|a_n| \leq \frac{14}{n-2} \to 0$$

4. What about

$$a_n = \frac{n^3 + n^2 - 6n + 7}{n^3 - n^2 - n + 3}?$$

Here we expect the limit to be $\frac{n^3}{n^3} = 1$. Let $b_n = a_n - 1$.

$$b_n = \frac{(n^3 + n^2 - 6n + 7) - (n^3 - n^2 - n + 3)}{n^3 - n^2 - n + 3} = \frac{2n^2 - 5n + 4}{n^3 - n^2 - n + 3}$$

and we see that $|b_n| \to 0$.

Home work.

For the following sequences examine if the limit exist as $n \to \infty$? If it does what is it? Give your reasons.

1. $a_n = \frac{n+\sin n}{n^2}$
2. $a_n = \sin n\pi$
3. $a_n = 1 + (-1)^n 10^{-8}$
4. $a_n = n^2$
5. $a_n = \frac{1}{\log n}; n \geq 2$