September 16

Free Tutorial service. Students can receive free tutoring from their peers in the Tutoring Center (Warren Weaver Hall, Room 524). The hours of operation are listed here:

$$http://math.nyu.edu/degree/undergrad/tutor\_schedule.html$$

Improper Integrals. Trying to integrate $\int_0^1 \frac{dx}{x}$ leads to problems.

$$\int_0^1 \frac{dx}{x} = \log x|_0^1 = \log 1 - \log 0 = -\log 0 = ?$$

Does not exist. But for small $\epsilon > 0$, the integral

$$\int_\epsilon^1 \frac{dx}{x} = \log 1 - \log \epsilon = \log \frac{1}{\epsilon}$$

and $\log \frac{1}{\epsilon}$ becomes large as $\epsilon \to 0$. We say the integral diverges. Area under the curve $y = \frac{1}{x}$ between $x = 0$ and $x = 1$ is infinite. You can see it by examining the rectangles on $[2^{-k}, 2^{-k+1}]$ with height equal to 2^{k-1} which are disjoint and all below the curve $y = \frac{1}{x}$. Area of any of the rectangles is $2^{-k} \times 2^{k-1} = \frac{1}{2}$. Each rectangle has area $\frac{1}{2}$ and there are an infinite number of them . So the total area below the curve, which is at least as much as the sum of the areas of all the rectangles, is infinite (i.e larger than any finite number).

We can not attribute it simply to the curve $y = \frac{1}{x}$ becoming unbounded at $x = 0$. In fact $y = x^{-p}$ has similar behavior for all $p > 0$. If $p \neq 1$, then

$$\int_\epsilon^1 x^{-p} dx = \frac{1}{1-p} x^{-p+1}|_0^1 = \frac{1-\epsilon^{1-p}}{1-p}$$

It becomes large as $\epsilon \to 0$ if $p > 1$ and has a nice limit $\frac{1}{1-p}$ if $p < 1$. This improper integral converges if $p < 1$ and diverges if $p > 1$. We saw it diverges also when $p = 1$.

There are similar problems when we want to integrate over the whole line $[0, \infty)$.

Let $a > 0$

$$\int_0^\infty a\,dx = ax|_0^\infty = \infty?$$

Well,

$$\int_0^L a\,dx = aL$$

and $aL \to \infty$ as $L \to \infty$. Therefore, for $\int_0^\infty f(x)\,dx$ to have a chance of being finite $f(x)$ has to be small near ∞. But that is not enough. For $p \neq 1$,

$$\int_0^L \frac{1}{(1+x)^p}\,dx = \frac{1}{1-p}[L^{1-p} - 1]$$

As $L \to \infty$, for $p > 1$ the limit is $\frac{1}{p-1}$. If $p < 1$ the limit is infinite.

If the function that is being integrated is non-negative then as the interval of integration increases the value can only go up. the question then boils down to, is it increasing with out bounds . i.e tending to infinity or is it increasing to a finite limit? An increasing function cannot oscillate. On the other hand

$$\int_0^L \cos x \, dx = \sin L$$

oscillates between -1 and 1 and does not converge, although it does not become large.

If $0 \leq f(x) \leq g(x)$ and the integral $\int_a^b g(x)\, dx$ is convergent so is the integral of $f(x)$. The integral of g remains bounded and the integral of f is smaller and therefore bounded as well. Since the integrand is non-nengative the integral of f converges. This can be used in many ways. Actually $f(x) \leq Cg(x)$ for some constant C is enough. In particular if at the "bad point" a for two non-negative functions f and g,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = c \neq 0$$

then either both integrals converge or both diverge.

$$\int_0^1 \frac{1 - \cos x}{x^2} dx < \infty$$

$$\int_1^\infty \frac{1 - \cos x}{x^2} dx < \infty$$

Note that $\frac{1-\cos x}{x^2}$ has a limit $\frac{1}{2}$ as $x \to 0$ and so there is no problem at 0. On the other hand $(1 - \cos x) \leq 1$ and $\int_1^\infty \frac{dx}{x^2}$ converges because we have $\frac{1}{x^p}$ with $p = 2 > 1$.

Home work. Do the following integrals, converge, diverge or oscillate?

$$\int_0^1 \log x \, dx$$

$$\int_0^1 \frac{1}{x \log x} \, dx$$

$$\int_0^1 \frac{\sin x}{x} \, dx$$

$$\int_0^1 \frac{\sin x}{x^2} \, dx$$

$$\int_0^\infty x^{10} \, e^{-x} dx$$