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Integration of trigonometric polynomials.

To integrate

$$\int sin^m x \cos^n x \, dx$$

When one of them (either m or n) is odd. Let $n = 2k + 1$ be odd Put $u = sinx$.

$$\int u^m (1 - u^2)^k du$$

Let both both be even. $m = 2k$ and $n = 2\ell$. Use $\sin^2 x = \frac{(1-\cos 2x)}{2}$ and $\cos^2 x = \frac{(1+\cos 2x)}{2}$

$$\frac{1}{2^{\ell+k}} \int (1 - \cos 2x)^k (1 + \cos 2x)^\ell dx$$

Simplify. Put $y = 2x$. Various powers of $\cos y$ show up. Odd powers can be handled. Even powers use half angle formula. Cut down the power by a factor 2. Repeat till done.

Example 1.

$$\begin{aligned}\int \sin^2 x \cos^2 x dx &= \frac{1}{4} \int (1 - cos2x)(1 + \cos 2x) \, dx \\ &= \frac{1}{4} \int (1 - \cos^2 2x) dx = \frac{x}{4} - \frac{1}{8} \int (1 + \cos 4x) \, dx \\ &= \frac{x}{4} - \frac{x}{8} - \frac{1}{32} \sin 4x + C = \frac{x}{8} - \frac{1}{32} \sin 4x + C\end{aligned}$$

Example 2.

$$\begin{aligned}\int e^{ax} \sin bx \, dx &= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx \\ &= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx \, dx\end{aligned}$$

$$(1 + \frac{a^2}{b^2}) \int e^{ax} \sin bx \, dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2} e^{ax} \sin bx$$

$$\int e^{ax} \sin bx \, dx = \frac{1}{a^2 + b^2} [-b \, e^{ax} \cos bx + a \, e^{ax} \sin bx] + C$$

Example 3. Integrating

$$\int \frac{dx}{a \, x^2 + bx + c}$$

Depends on the roots of the quadratic $a \, x^2 + bx + c$. Two distinct real roots.

$$a \, x^2 + bx + c = a(x - \alpha)(y - \beta)$$

Then partial fraction works. If we have two coincident roots, it is a square. Becomes

$$\int \frac{dx}{a(x-\alpha)^2}dx$$

If no real roots it becomes by completing the square

$$a[(x-\alpha)^2+\beta^2]$$

Substituting $x=\alpha+\beta y$ this reduces to

$$\frac{1}{a\beta}\int \frac{dy}{1+y^2} = \arctan y + C$$

Other things to remember from calculus I.

What is $\sqrt{100.1}$ =?. 10.005
It is nearly 10. Say $10+x$. Then

$$(10+x)^2 = 100+20x+x^2 = 100.1$$

$$20x+x^2 = 0.1; \quad x \simeq \frac{.1}{20} = .005$$

For more accuracy try $10.005+x$. Then

$$10.005^2 + (20.01)x + x^2 = 100.1$$

Ignore x^2. Next approximation is $10.005 + \frac{100.1-(10.005^2)}{20.01} = 10.00499875$

Home work Problems: Calculate the Integrals.

$$\int_0^1 x^{\frac{3}{2}}(1-x)^2 dx$$

$$\int_0^{\frac{\pi}{2}} (\sin x)^4 (\cos x)^4 \, dx$$

$$\int_0^1 xe^{2x}dx$$

$$\int_0^1 \frac{dx}{x^2+4}$$

$$\int_0^1 \frac{dx}{x^2-4}$$