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Conic Sections

Conic sections consist of three classes of curves. Ellipses, Hyperbolas and Parabolas.

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represents an ellipse. It is like an oval, with an axis along the x axis of length $2a$ and an axis along the y axis of length $2b$. If $a = b$ it is a circle with radius a. Its total area is $\pi\, a\, b$. The circumference is hard to calculate.

If P Q are two points in the plane and a point moves keeping the sum of the distances from P and Q a constant, then it describes an ellipse. To see this, Take the points as $(-a, 0)$ and $(a, 0)$. If (x, y) is the moving point, and the sum of the distances for P and Q of the moving point is $2\ell > 2a$, then

$$\sqrt{(x+a)^2 + y^2} + \sqrt{(x-a)^2 + y^2} = 2\ell$$

After squaring

$$(x+a)^2 + (x-a)^2 + 2y^2 + 2\sqrt{(x+a)^2 + y^2}2\sqrt{(x-a)^2 + y^2} = 4\ell^2$$

Bringing the square root term to one side and the rest to he other side

$$(x+a)^2 + (x-a)^2 + 2y^2 - 4\ell^2 = -2\sqrt{(x+a)^2 + y^2}2\sqrt{(x-a)^2 + y^2}$$

After squaring both sides

$$[(x+a)^2 + (x-a)^2 + 2y^2 - 4\ell^2]^2 = 4[(x+a)^2 + y^2][(x-a)^2 + y^2]$$

or

$$[2x^2 + 2y^2 + 2a^2 - 4\ell^2]^2 = 4[x^2 + y^2 + a^2]^2 - 16a^2x^2$$

After dividing by 4,

$$[x^2 + y^2 + a^2 - 2\ell^2]^2 = [x^2 + y^2 + a^2]^2 - 4a^2x^2$$

Expanding the squares, rearranging the terms and dividing by 4,

$$\ell^4 - \ell^2(x^2 + y^2 + a^2) = -a^2x^2$$

or

$$(\ell^2 - a^2)x^2 + \ell^2y^2 = \ell^2(\ell^2 - a^2)$$

or

$$(\ell^2 - a^2)x^2 + \ell^2 y^2 = \ell^2(\ell^2 - a^2)$$

or

$$\frac{x^2}{\ell^2} + \frac{y^2}{\ell^2 - a^2} = 1$$

It is an ellipse. $(0, 0)$ is the center. $(\pm a, 0)$ are called the foci. The major axis has length 2ℓ and the minor axis length $2\sqrt{\ell^2 - a^2}$

Now if P Q are two points in the plane and a point moves keeping the *difference* of the distances from P and Q equal a constant $2\ell < 2a$, then it describes a hyperbola. To see this, Take the points as $(-a, 0)$ and $(a, 0)$. If (x, y) is the moving point, and the sum of the distances for P and Q of the moving point is $2\ell > 2a$, then

$$\sqrt{(x+a)^2 + y^2} - \sqrt{(x-a)^2 + y^2} = 2\ell$$

After squaring

$$(x+a)^2 + (x-a)^2 + 2y^2 - 2\sqrt{(x+a)^2 + y^2}2\sqrt{(x-a)^2 + y^2} = 4\ell^2$$

Bringing the square root term to one side and the rest to he other side

$$(x+a)^2 + (x-a)^2 + 2y^2 - 4\ell^2 = +2\sqrt{(x+a)^2 + y^2}2\sqrt{(x-a)^2 + y^2}$$

After squaring both sides

$$[(x+a)^2 + (x-a)^2 + 2y^2 - 4\ell^2]^2 = 4[(x+a)^2 + y^2][(x-a)^2 + y^2]$$

Now it is no different from the previous case. End up with the same equation.

$$\frac{x^2}{\ell^2} + \frac{y^2}{\ell^2 - a^2} = 1$$

Except now $\ell < a$ and so we need to write it as

$$\frac{x^2}{\ell^2} - \frac{y^2}{a^2 - \ell^2} = 1$$

which makes it a hyperbola.

A point P moves so that its distance from a line is proportional to the distance from a point. Take the point to be the origin $(0, 0)$ the line to be $x = a$. Then

$$x^2 + y^2 = c^2(x-a)^2$$

or

$$x^2(1 - c^2) + y^2 + 2ac^2 x = c^2 a^2$$

or

$$(1-c^2)(x-a\frac{c^2}{1-c^2})^2+y^2=c^2a^2+\frac{c^4}{1-c^2}=\frac{c^2}{1-c^2}[c^2+a^2-a^2c^2)]$$

We rewrite it as

$$\frac{(x-x_0)^2}{A^2}+\frac{y^2}{B^2}=1$$

with $x_0=\frac{ac^2}{1-c^2}$, $A^2=\frac{c^2}{(1-c^2)^2}[c^2+a^2-a^2c^2)]$ and $B^2=\frac{c^2}{(1-c^2)}[c^2+a^2-a^2c^2)]$ It is an ellipse if $c<1$, a hyperbola if $c>1$ and a parabola if $c=1$. The line is called the directrix and the point a focus. c is called the eccentricity and is usually denoted by e. The circle is the limiting case when $c=0$ and $a=\infty$. The center of the ellipse is located at $(\frac{ac^2}{1-c^2},0)$. Its major i.e the larger axis is

$$\frac{c}{1-c^2}\sqrt{c^2+a^2(1-c^2)}$$

and the minor axis is

$$c\sqrt{\frac{c^2}{1-c^2}+a^2}$$

If $a\to\infty$ and $c\to 0$ with $ac\to r$ we get a circle with radius r.

To determine the focus and directrix of an ellipse, we want to write

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with $a>b$ in the form

$$(x-x_0)^2+y^2=e^2(x-d)^2$$

$$\frac{b^2x^2}{a^2}+y^2=b^2$$

identify terms.

$$e^2d^2-x_0^2=b^2,\qquad 1-e^2=\frac{b^2}{a^2},\qquad e^2d=x_0$$

then $(x_0,0)$ will be the focus, e the eccentricity and $x=d$ the directrix. Solve for x_0,e and d.

$$e=\sqrt{1-\frac{b^2}{a^2}}$$

$$e^2d^2-e^4d^2=b^2,\quad d^2=\frac{b^2}{e^2(1-e^2)}=\frac{a^2}{e^2},\quad d=\frac{a}{e}$$

$$x_0=e^2d=ae$$

The equation is rewritten as

$$(x-ae)^2+y^2=e^2(x-\frac{a}{e})^2$$

which is the same as

$$(x+ae)^2+y^2=e^2(x+\frac{a}{e})^2$$

showing that there is another pair of focus-directrix on the other side.

Let us look at some examples.

$$\frac{x^2}{4}+y^2=1$$

$a=2, b=1, e^2=1-\frac{1}{4}=\frac{3}{4}, e=\frac{\sqrt{3}}{2}$. Focus is $(ae,0)=(\sqrt{3},0)$ Directrix is $x=\frac{4}{\sqrt{3}}$. Rewrite the equation as

$$(x-\sqrt{3})^2+y^2=\frac{3}{4}(x-\frac{4}{\sqrt{3}})^2$$

In polar coordinates with origin as focus

$$r=\sqrt{3}((\frac{4}{\sqrt{3}}-\sqrt{3})-(x-\sqrt{3}))=\frac{\sqrt{3}}{2}(\frac{1}{\sqrt{3}}-(x-\sqrt{3}))=\frac{\sqrt{3}}{2}(\frac{1}{\sqrt{3}}-r\cos\theta)$$

$$r=\frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}\cos\theta}$$

The case of the hyperbola is almost the same. b^2 gets replaced by $-b^2$. So the only change is $e=\sqrt{1+\frac{b^2}{a^2}}$.

A hyperbola has asymptotes. The lines $x\pm y=0$ are the asymptotes for $x^2-y^2=1$. Comes from the fact that

$$\lim_{x\to\infty}[\sqrt{x^2-1}-x]=0$$

$$\lim_{x\to\infty}\sqrt{x^2-1}-x=\lim_{x\to\infty}x\left[\sqrt{1-\frac{1}{x^2}}-1\right]=\lim_{y\to 0}\frac{\sqrt{1-y^2}-1}{y}=0$$

by L'Hospital's rule. The pair of asymptotes to

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0; \quad \text{i.e} \quad \frac{x}{a}\pm\frac{y}{b}=0$$

For the parabola with focus at $(0,0)$ and directrix $x=a$, the eccentricity $c=1$ and the equation is

$$x^2+y^2=(x-a)^2$$

or

$$y^2 = a^2 - 2ax = 2a(\frac{a}{2} - x)$$

The parabola $x = -y^2$, is rewritten as

$$y^2 = 2 \cdot \frac{1}{2}(\frac{1}{4} - (x + \frac{1}{4}))$$

making the directrix $x = \frac{1}{2}$ and the focus $(-\frac{1}{4}, 0)$.

The equations of conic sections are just as easy to write in polar coordinates. If the origin is the focus and the directrix is the line $r \sin\alpha \cos\theta + r\ \cos\alpha\ \sin\theta = p$, and the eccentricity is e, the polar equation becomes

$$r = e|r \sin\alpha \cos\theta + r\ \cos\alpha \sin\theta - p|$$

or

$$r = \frac{c}{1 - a\sin\theta - b\cos\theta} = \frac{c}{1 - e\cos(\theta - \alpha)}$$

c determines the size, α the orientation and e is the eccentricity, After rotation

$$r = \frac{c}{1 - e\cos\theta}$$

$c = ep$ where p is the distance from the focus. $e = 0$ is the circle. $e > 1$ hyperbola. What about $e = 1$?

$$r(1 - \cos\theta) = c; \quad r = (x + c); \quad x^2 + y^2 = x^2 + 2cx + c^2; \quad y^2 = 2c(x + \frac{c}{2})$$

parabola with directrix $x = -c$ and focus at $(0, 0)$ and vertex at $(-\frac{c}{2}, 0)$.

Examples. 1. $r = \frac{2}{3 - \sin\theta}$

$$r = \frac{\frac{2}{3}}{1 - \frac{1}{3}\sin\theta}$$

$e = \frac{1}{3}$. Ellipse.

$$r = \frac{2}{3} + \frac{1}{3}r\ + 3\sin\theta = \frac{1}{3}(y + 3)$$

Directrix is $y = \frac{-1}{3}$. Focus at $(0, 0)$. $e = \frac{1}{3}$. The major axis is along the y axis.

2. $r = \frac{2}{3 - \cos\theta}$

$$r = \frac{\frac{2}{3}}{1 - \frac{1}{3}\cos\theta}$$

$e = \frac{1}{3}$. Ellipse.

$$r = \frac{2}{3} + \frac{1}{3}r \ + 3\cos\theta = \frac{1}{3}(x+3)$$

Directrix is $x = \frac{-1}{3}$. Focus at $(0,0)$. $e = \frac{1}{3}$. The major axis is along the x axis.

3. $r = \frac{2}{2-3\cos\theta}$

$$\frac{2}{2-3\cos\theta} = \frac{1}{1-\frac{3}{2}\cos\theta}$$

$e = \frac{3}{2} > 1$. Hyperbola. Focus at $(0,0)$.

$$r = 1 + \frac{3}{2}r\cos\theta = \frac{3}{2}(\frac{2}{3} + x)$$

Directrix is $x = -\frac{2}{3}$. The asymptotes are determined by $r = \infty$. $\cos\theta = \frac{2}{3}$.

Homework.

For the following conic sections in cartesian coordinates determine if it is an ellipse, parabola or a hyperbola. Determine the eccentricity, identify a focus and the corresponding directrix and write the equation in polar coordinates shifting the origin to the focus.

Q1. $x^2 - y^2 = 1$

Q2. $y^2 - x^2 = 1$

Q3. $(3x+2y)^2 = (2x+3y)^2 + 5$

For the following conic sections in polar coordinates determine if it is an ellipse, parabola or a hyperbola. Determine its eccentricity, Describe it in Cartesian coordinates in the form

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = \pm 1$$

or

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = \pm 1$$

or

$$\frac{(y-y_0)^2}{b^2} - \frac{(x-x_0)^2}{a^2} = \pm 1$$

Q4. $r = \frac{3}{4-\sin\theta}$

Q5. $r = \frac{3}{1-4\cos\theta}$